DOCUMENT RESUME

ED 372 931 SE 054 248

AUTHOR Kupari, Pekka, Ed.; Haapasalo, Lenni, Ed.
TITLE Constructivist and Curriculum Issues in School Mathematics Education: Mathematics Education Research in Finland: Yearbook 1992-1993. Theory into Practice 82. Publication Series B.
INSTITUTION Finnish Association of Mathematics and Science Education Research.; Institute for Educational Research in the Netherlands (SVO), The Hague.
REPORT NO ISBN-951-34-0180-4; ISSN-0782-9817
PUB DATE 93
NOTE 125p.
AVAILABLE FROM Institute for Educational Research, University of Jyvaskyla, PO Box 35, SF-40351, Jyvaskyla, Finland (70 finmark; native language version available).
PUB TYPE Collected Works - General (020)

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS *Constructivism (Learning); Deduction; Difficulty Level; Educational History; Elementary Secondary Education; Epistemology; Foreign Countries; Heuristics; *Learning Processes; Mathematics Curriculum; *Mathematics Education; *Student Evaluation; *Teaching Methods; Test Items; *Thinking Skills
IDENTIFIERS *Alternative Assessment; Finland

ABSTRACT

This yearbook is the seventh for the Mathematics Education Research in Finland series and contains ten articles presented at the decennial jubilee colloquium. The first three articles discuss Constructivism both in general and with regard to mathematics education. Analysis is directed to the origins of constructivism, to the learning of mathematical concepts within systematic constructivism, and to assessment within a social constructivist framework. The fourth article analyzes the possibilities for deductive reasoning from the perspective of cognitive psychology, and the fifth discusses how to promote the mediation of thinking skills and heuristic processes of an expert in a complex technological environment. The next two articles discuss change both in the mathematics curriculum and in mathematics teaching. The sixth article describes the curriculum work in the Finnish comprehensive school during the last few years and the seventh article asks "Is school teaching changing?". The last three articles emphasize epistemological perspectives and historical sources in creating a new approach to the teaching of mathematics, give an outline of the history of mathematical contests in Estonia, and present an approach for predicting the level of difficulty of mathematical items. (MKR)

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INSTITUTE FOR EDUCATIONAL RESEARCH. PUBLICATION SERIES B

ED 372 931

Pekka Kupari – Lenni Haapasalo (Eds.)

Constructivist and Curriculum Issues in School Mathematics Education

Mathematics Education Research in Finland: Yearbook 1992–1993

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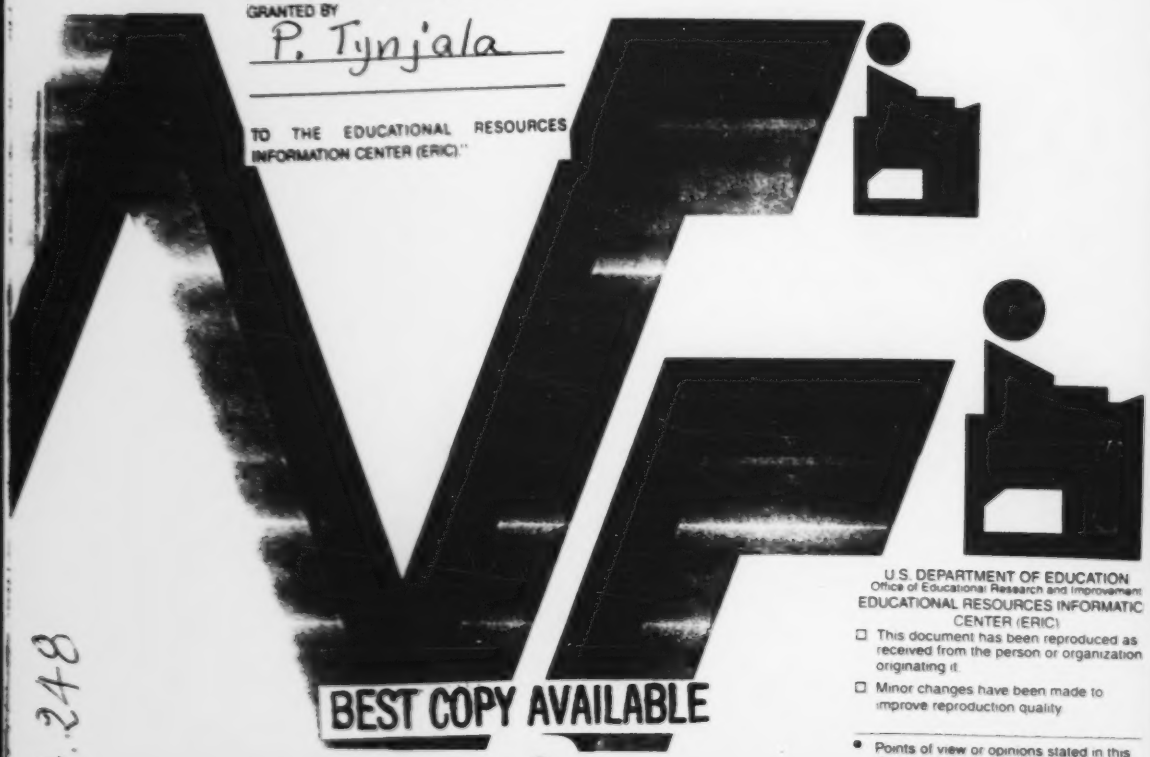

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SE 054 248

YEARBOOK 1992–1993

Published by the Finnish Association
of Mathematics and Science Education Research
and the Institute for Educational Research

Edited by Pekka Kupari & Lenni Haapasalo
Price of 70 Fmk
Address and orders: University of Jyväskylä
Institute for Educational Research
PB 35
FIN - 40351 Jyväskylä
Finland
Printed in Jyväskylän Yliopistopaino
Cover: Hetimonex kirjapaino

SE

Institute for Educational Research. Publication Series B. Theory into Practice 82.

Constructivist and Curriculum Issues in School Mathematics Education

Mathematics Education Research in Finland: Yearbook 1992-1993

Edited by Pekka Kupari & Lenni Haapasalo

Institute for Educational Research
The Finnish Association of Mathematics and Science Education Research

Jyväskylä 1993

ISBN 951-34-0180-4
ISSN 0782-9817

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Institute for Educational Research and the
Finnish Association of Mathematics and Science
Education Research

Jyväskylän Yliopistopaino
Kirjapaino Hetimonex - Jyväskylä 1993

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Kupari, P. & Haapasalo, L. (Eds.) 1993. Constructivist and curriculum issues in school mathematics education. Mathematics education research in Finland. Yearbook 1992-1993. University of Jyväskylä. Institute for Educational Research. Publication series B. Theory into Practice 82.
ISBN 951-34-0180-4
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The Yearbook 1992-1993 is the seventh yearbook and it includes ten articles. The first three articles discuss constructivism both in general and especially with regard to mathematics education. Analysis is directed to the origins of constructivism, to learning of mathematical concepts within "systematic constructivism" and to assessment within a social constructivist framework. The fourth article analyzes the possibilities for deductive reasoning from the perspective of cognitive psychology, and the fifth discussess how to promote the mediation of thinking skills and heuristic processes of an expert in a complex technology environment. The following two articles discuss change both in the mathematics curriculum and in mathematics teaching. The sixth article describes the curriculum work in the Finnish comprehensive school during the last few years and the seventh article asks "Is school teaching changing?". The last three articles are based on the presentations of the foreign guest speakers in our 1993 national colloquim. They emphasize epistemological perspectives and historical sources in creating a new approach to the teaching of mathematics, give an outline of the history of mathematical contests in Estonia and present an approach to predict the level of difficulty of mathematical items.

Descriptors: mathematics, constructivism, curriculum, teaching, assessment, thinking, reasoning, history of mathematics

PREFACE

In 1982 the Finnish Association of Mathematics and Science Education Research (FAMSER) arranged in Jyväskylä its first national colloquim that has since become an annual event. This seventh yearbook of the Association consists of the presentations given in the decennial jubilee colloquim in Jyväskylä in September 1992.

The main themes of the colloquim were mathematics and science curricula in general education and the place of these subjects in teacher education. An essential role of the presentations was in the constructivist view of learning that can clearly be seen also in the development of the Finnish mathematics education and has to some extent influenced curriculum work of all school subjects. Mathematics has been a predecessor in the reform process of subject curricula, because the work in the field of mathematics on defining the guidelines for the development of teaching was started in the beginning of 1990.

This yearbook is probably the last to be published by the Association. The reasons for discontinuing the publishing are both economic and scientific. A Scandinavian publication, NOMAD (Nordisk Matematikk Didaktikk i.e. Nordic Studies in Mathematics Education) has lately been founded and its first issue came out in October 1993. For example, due to the birth of this new publishing forum it is appropriate that we try to channel the results of the Finnish research of mathematics and science education to more well-known and more widely spreading publications and publication series.

Pekka Kupari *Lenni Haapasalo*

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Origins and Varieties of Constructivism

Jarkko Leino
University of Tampere

Constructivism is not a single theory but a variety of theories, an approach or a paradigm. It is often applied in education and psychology in some form but it has its roots also in theory of science as well. In the article several varieties of constructivism are dealt with as a background of mathematics education.

1. Introduction

Constructivism is often dealt with in a rather simplistic way in order to give a teacher, interested in the matter, more meaningful and student centred basis for teaching than the so called behaviouristic theory (e.g. Leino 1989). Constructivism is then viewed according to its name as an approach in which a student actively constructs his (her) knowledge on the basis of his previous understanding. Constructivism is, however, much more. When analysed at greater depth, several different variations and origins can be found in which the essence and construct of knowledge have to be taken seriously. I will try to describe these different constructivisms within the frame of mathematics teaching.

2. Naive constructivism

If we constrain our focus on the way an individual student actively constructs his knowledge base, without any other assumptions, we can speak of naive or weak constructivism. To my mind even this form of constructivism means a big step forward in making teaching and learning process active and seeing knowing and knowledge as problematic. Knowledge is not any more just transferable information, something to be given and received, and learnt or not as such, but it is instead something individual and personal. Because learning a new matter or forming a new conception of something already learnt always changes the knowledge structure of the learner, which means that learning is a recursive activity. Previous experiences form the basis and simultaneously set the limits. Looking at the very heart of this process we have to ask how the first experiences

are formed and what are the child's or adult's constraints, which prevent him from accessing the knowledge regarded as useful and even necessary to his development. These questions are very difficult.

Learning as information processing can be included in naive constructivism because the information processing approach, too, considers the human being as active and learning as a constructive process monitored by certain strategies. Becoming conscious of one's own strategies, choosing a suitable one and organizing the use of memory can be described as metacognitions. These are, however, to be regarded as only irrelevant details within the main idea. If the so-called external world is considered an unproblematic and objective reality, as most representatives of cognitive psychologists do, the limits of naive constructivism are not broken in any way. The truth of knowledge is then based on its correspondence with reality and the criterion for this is human perception. Of course, there are problems if one asks where the child's cognitions and strategies come from, if something is given in birth or if all cognitions come through perceptions and activites, how strategies develop, and if and how it is possible to influence them by teaching.

Though constructivism can be given deeper meaning by adding assumptions e.g. on the concept of knowledge, this does not mitigate the fact that seeing the human being as an active information processor has strongly changed the theories of learning and teaching, especially from the point of view of research. The information processing approach has started a new research direction in which modelling human knowing and knowledge construction have been emphasized and been the object of systematic theory formation for the needs of teaching. This has meant a crucial change in many aspects of learning and teaching in research as well as in practice.

3. Piaget's constructivism

Studies on mathematics education very often refer to Piaget's theories and constructivism. However, Piaget's interest in knowledge was basicly epistemological and followed the ideas of Kant. In this approach the human being can not perceive the reality objectively, such as it is, but only within the constraints of his previous experiences and knowledge construction. Thus, Piaget had already the essential factors of the so called strong or radical constructivism. The child's external world is not necessarily similar to the adult's, neither do individuals perceive their environment similarly at different times.

Piaget emphasized the meaning of the child's actions for the development of thinking and concept forming. In the background, there is a kind of individualistic-

Darwinian conception of an organism's changes in order to adapt to its environment. An organism interacts with its envcironment and constructs its cognitive structure within the limits of its genetic possibilities and constraints. It is seen as a mechanism which inquires and uses information and changes itself simultaneously. Through its actions the organism learns to recognize the objects in the environment, makes them useful and constructs the internal representation of the external world. In order to know objects, the subject has to act upon them, and therefore transform them: displace, connect, combine, take apart and reassemble. This means two things: the subject has to coordinate his actions with the environment and to learn the interactions between him and the objects of the reality. These two forms of activities are interdependent because it is only through action that these relations originate. Through these actions the subject forms a scheme of a permanent object, which becomes established independent of the subject's own actions. Piaget used two kinds of schemes: scheme (plural schemes) was used to refer to operational activities, whereas schema (plural schemata) referred to the figurative aspects of thought, attempts to represent reality without attempting to transform it. In particular, based on the operational schemes Piaget created his theory of qualitatively different and hierarchial levels of development of man. The child learns in qualitatively different ways on different levels independent of the topic. This theory has frequently been applied in teaching, especially in natural sciences. (see e.g. Piaget 1988.) Later on, studies have mainly shown these differences between the levels to be caused by the research methods, not qualitatively different to a remarkable extent and not independent of the topics (e.g. Meadows 1988, Keil 1988).

Piaget's studies do not penetrate into the issue of how the child's thinking develops in respect of the environment but into the relationship between the child's perceptions and conceptions. Because perception cannot be equated with the environment, it is very difficult to interpret and apply Piaget's studies for teaching. They are quite far from the actual needs of e.g. producing instructional materials or other similar factors in students' learning environment.

von Glasersfeld seems to have a leaning to Piaget's theories but he emphasizes in them the deficiency of an individual to form true knowledge of the external reality, and consequently, the unavoidable adaptation to viable knowledge, to knowledge that seems to work according to the individual's experiences. This form of constructivism easily provokes an opposition particularly among mathematicians because it takes truth as a unachieveable dream: the function of cognition is only adaptive and serves the organization of the experiential world and not the discovery of ontological reality. This is what makes many scholars, who have constructivism as their theoretical view, remain naive (or realistic) constructivists

rather than radical. It is worth mentioning that radical constructivists do not deny the existence of external reality but believe that the truth of knowledge cannot be achieved and, hence, we have to contend with viable knowledge, consistent with our experiences. (von Glasersfeld 1990.)

4. Social constructivism

Man's external world does not consist of only the physical but the social world as well, and, hence, of the reliable knowledge man has gathered during his historical development. Radical constructivism regards also this kind of knowledge as being constructed again and again by an individual and only within the limits of his experiencs, not necessarily such as presented. The concept of number, for instance, is only an individual's personal conception depending on the time and the stage of his development, not necessarily the one mathematicians have created. In the same way the social values or shared feelings are seen as nothing but subjective states, competing versions of the law of the jungle (Ernest 1992). Should education, in particular, be more serious and permanent, was Ernest's question.

In social constructivism human beings are regarded as indissolubly interconnected. Human beings are formed through their interactions with each other and cannot be considered as wholly isolated. An individual mind is part of a broader context, the social construction of meaning. Social meanings are commonly constructed and shared but modified all the time and interacting to fit to previous ontological reality. The human communication bases on language which is basicly metaphorical. Language carries with it the previous shared meanings which change along people's functions. Shared social context is the basis of thinking but simultaneously it works as a constraint of autonomy and possibilites.

There are several constructivisms in education, as mentioned above, and, hence, constructivism cannot be regarded as a theory. Rather it is an approach, metatheory or paradigm. However, there are some common assumptions of human beings in all of them:

(1) Human being can learn only in respect to his previous knowledge which has important consequences for how he learns and acts. This knowledge has a complex set of referents and meanings that must also be taken into account when learning is studied.

(2) The learner has, at least partly, a control over his learning, though several social and factual norms constrain this autonomy. One of the consequences of this assumption is that learning and teaching has to

be understood as being constructed purposively by the learner and teacher themselves, and not to be studied without these purposes.

(3) Human being has developed capacity for developing knowledge by organizing complexity rapidly, attending to the meanings of complex communications and behaving within the social norms. (Magoon 1977.)

The third assumption emphasizes the inborn, a priori forms of time and space, already presented by Kant, which make it possible for the human being to develop socially but which also set certain limits to autonomy.

Marton and Neuman (1992) viewed constructivism at ICME 7 via phenomenology. According to them, the world we live in is always seen by us as a world which is experienced, understood in one way or other, not independent of us, neither are we independent of the world. When we are studying how certain mathematical elements or other objects are experienced we study the very aspects of the world, as they appear to us, as much as we study acts of experiences. Knowledge is the sum of our experiences. Reflecting on knowledge is reflecting on what is experienced. Learning is said to occur when there is a change in what is experienced. Research on learning always concerns different experiences expressed in one and the same problem-solving act. The phenomenological researcher is interested in those different meanings students have of e.g. the number facts.

5. Constructivism as theory of science

Researchers on learning and teaching form concepts for dealing with phenomena and certain connections between the concepts in order to build theories. Though there exist several approaches to this, constructing a theory is based on some common norms. The same happens in other fields of sciences. Constructivism can also be understood as a broader theory of science than the above mentioned versions of constructivism, with the main aspects dealing with how the concepts and theories are constructed and how to work within the norms set for this, no matter whether they are genotypical or social.

As a theory of science constructivism is, in mathematics, much more and older than cognitive psychology or Piaget's studies and theories of learning. In fact, the reasons for the debate of the foundations of mathematics penetrate more deeply and more commonly into the essence and the tasks of science than is usually thought of when speaking of learning. We can say that the debate of the existence of mathematical elements already brought constructivism into the philosophy of mathematics: constructivism fought against the so called formalism according to which e.g. arithmetics could be interpreted as a calculus, based on certain evidence

and indisputable axioms. The absence of contradiction in the axiom system was not enough to constructivism, for the existence should be fit to the demand of being constructed. For example, arithmetics is not just an axiomatic science but numbers, which form the foundation, should be created by constructing the operations, consistent with certain norms. The constructivist demand is extended to all sciences as a basic demand for methodical thinking. All argumentations should be gradually and non-circularly based on the norms, and only the concepts constructed in this way are acceptable in science.

The language in the constructivist theory of science has to follow the rules in all argumentations. The meanings of natural language cannot be used at any stage of argumentation but instead through pretheoretical understanding one attempts to search methodologically formed understanding of the connections in the world of phenomena. It is not my purpose to penetrate into the finesses of the so called Erlangen school (see Lorentzen 1974, in Finnish Töttö 1989) but I will only briefly state that the constructivist theory of science emphasizes certain norms being always on the basis of any saying and understanding of an argumentation. Actions, language and norms posses a central position in all dialogues. For instance, the natural sciences, too, are based on certain norms because the objectivity of measuring is based on agreements of the ideal demands on measuring. In the same way we can try to understand the logic in speaking and acting by reconstructing it methodologically. To my mind this is not, speaking of ideals, very far from the demands teachers and researchers face when attempting to construct students speaking or acting in mathematical contexts.

Constructivist theory of science is close to critical science or the so called postmodernism in the demand of methodological reconstruction. In postmodernism the teacher has to rethink of e.g. the contents of curriculum and ask of their relevance and importance to the learner (cf. Aronowits & Giroux 1991). Posmodernism also takes into account the environment of a student with its cultural-historical aspects. Some scholars name this approach as cultural constructivism (e.g. Scott et al. 1992). This offers a broader background than the constructivism of Piaget or cognitive psychologists by taking the social environment with its values and traditions into account. These aspects certainly are effective in students' as well as teachers' thinking. Cultural artifacts effect people's actions and are transmitted through them. Mathematical concepts, computers and other similar artifacts have been developed for certain purposes and they continue their lives in a way which may differ to a great extent from the original purposes.

6. Constructivist teaching

If we take as the goal in mathematics teaching to develop students' mathematical knowledge and, in particular, its meaningful uses in their living environment, it is much more difficult a task for the teacher than just knowing and being capable to present certain fixed mathematics, concepts, numerations or operations. The teacher has to take as the basis the students' previous experiences, at least within the topic to be dealt with, and in terms of these delineate the students' probable conceptual mapping and possible developmental routes in order to organize the opportunities to modify and complete their cognitive structures. Social constructivism emphasizes students' constraints to meaningfully experience e.g. mathematical concepts or operations if these do not include the essence of this culture. Ethnomathematics has taken that aspect as a special research interest.

In the same way a researcher should be able to delineate students' different conceptions of central mathematical concepts. This is especially what Marton's group has been doing for two decades. However, in addition to these descriptions a researcher has also to reflect on and experiment in practice the routes of how an individual student can promote his thinking and experiences the "right" meanings of the concepts as meaningful and worth learning. Very few researchers of e.g. Marton's "phenomenographical" group have reached this methodological level; Newman (1987) is a good exeption.

Though all teachers want to make their teaching meaningful to each student this seldom succeeds. I have, during last few years, attempted to find methodological alternatives to traditional teaching in order to give more space to the students' world of thinking. For this purpose I have recommended the so called project study or project method (Leino 1991). There are some bureaucrats who seem to wonder whether this method remains just another hobby-related activity with a romantic child centredness in focus or whether it means going back to the progressive school of the beginning of the century. Of course, this may happen but so is the case in "traditional" teaching too! Applying project method is quite demanding and needs a reflective practitioner. A technician is not enough but he might do if teaching means nothing but covering the whole textbook page after page (just like last year and the year before that!).

Project study or other teaching methods of active inquiry are not necessarily to be seen as a replication of the old progressive school. Firstly, the present school and conditions are not same as half of century ago. Secondly, the benefits of active inquiry methods are easily seen by every experimental teacher who is willing to reflect on and compare them with earlier results. Thirdly, if a teacher wants to emphasize a constructivist approach he needs methods for his students to express

their previous experiences in acting and speaking. Those students who have difficulties in learning have usually difficulties in verbalising their thinking. That is why working with some project gives the teacher a better tool to "know" about students' thinking, how they intend to start and what kind of previous experiences they connect with the task. Acts are often more revealing than words or short solutions of normal small exercises in mathematics textbooks.

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"Systematic Constructivism" in the Guidance and Analysis of Mathematical Concept Building

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The present article introduces the structure and presents some results of the MODEM project (Model Construction for Didactic and Empirical Problems of Mathematics Education), which investigates learning of mathematical concepts within "systematic constructivism". The process of concept building proved to be for the most pupils a long process in which the phases of orientation, definition, identification, production and reinforcement are systematically involved. These phases also form a good framework for planning learning environments and evaluation. In the subproject MODEM 1 the concept of the gradient of a straight line was investigated with eighth grade pupils, and in MODEM 2 -3 the learning of fractions and decimals was investigated in the fourth and fifth grade, respectively. All these investigations showed that the identification phase plays a central role in concept building, and seems to create the best and most pleasant learning environment for the forming of concept attributes. Nevertheless, concept understanding is most reliably measured by production tasks, which, however, are usually difficult for pupils. In the cognitive test pupils' results were poor in the identification as well as in the production tasks, but surprisingly good in contextual application tasks. The modes of information representation (verbal, graphic, symbolic) were found to have a different effect on the thinking processes of the pupils. Systematically, however, a production within the same form is significantly more difficult than a production across different kind of representations. The tasks containing verbal representation were difficult for the pupils, and were reliable for measuring of concept understanding. Pupils who were guided in concept building in a framework of "systematic constructivism" had significantly better results than pupils taught in a conventional way. The differences in the cognitive performances of boys and girls were smaller in the MODEM-group than in the comparison group, showing that the constructivist learning environments had a more positive effect on the boys' learning. The analysis of conventional textbooks and lessons showed that they focus on the surface properties of concepts especially on very stereotyped mechanical tasks. An analysis of the pupils' answers revealed that the pupils developed highly inadequate mental models in conventional teaching, even in respect of the basic attributes of the concepts. Pupils' opinions of constructivist learning environments were positive. The teachers confirmed the fact that perseverance, sense of purpose, and consistency in pupils' work had also clearly improved in other subjects.

1. Remarks on constructivism

The mathematics taught in school is based on age-old principles and does not contain much that has been discovered in this century. In spite of the fact that this knowledge is the result of long-term heuristic processes, mathematics has been taught as prestructured knowledge from textbooks. In constructivism it is more important to take into account the historical construction processes of mathematics than to preserve conventional contexts in textbooks. This approach accentuates the active mental processes of the pupil involved in trying to build a picture of reality and trying to change it in social interaction with others.

But how can we provide children with opportunities for these kinds of constructions? How can we be sure that the child learns mathematical concepts and structures, not only context oriented knowledge and routines? Because mathematics is a systematic construction itself, does it not give us more constructivist models than any learning psychology? These kinds of questions have been the basis of the *"systematic constructivism"* of the MODEM-project.

New waves of mathematics teaching in England and the USA contain constructivist ingredients as can be noticed in the NCTM standards (1989, 1991). These often appear in relating mathematics to situations in everyday life and stressing the meaning of language in learning situations. The theoretical discussion, in spite of its philosophical sophistication, does not to affect school work. It is very fashionable to plan very strongly context-oriented learning environments for giving pupils opportunities to obtain adequate mathematical information from everyday situations (Fig. 1). They can, for example, study a leaflet on nutrition and they have to plan a healthy breakfast (problem 1). It is patently obvious that in social interactions children can learn some percentages (procedures 1-2), perhaps use the concept "fraction" (concept 1) etc. We can choose a new problem again and hope that the children can construct more mathematical knowledge for his everyday life. But the main question remains: How can we be sure that the children learn mathematical *concepts* and *structures*?

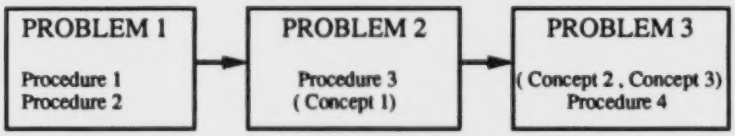

FIGURE 1. *Context-oriented learning of mathematics in "radical constructivism"*

It is very interesting to consider what kind of constructions the children can really make in these processes. One possibility is to interpret these kinds of learning environments within *radical constructivism* like von Glasersfeld (1991) by accepting the possibility that there does not need to exist any objective "mathematical reality" for the child. The only relevant interpretation is made by the pupil himself in the situation. This, however, contradicts the goal that the teacher has from the outset: the children must learn "advanced" mathematics that has some power in culture and everyday life. But if this is the case, then we should speak about *weak constructivism*, because the mental models (concept attributes) of the children should fit together reasonably precisely after the learning period.

Naturally it is reasonable to accept the possibility that the children develop certain social constructions of their own which can be used in communication. But it really is difficult to identify the borderline between

weak and radical constructivism.The situation is like a tennis tournament. The external global terms mean that we know when and where the final is going to be played. We can even make assumptions about tactics to be used by the players, if the final is, for example, played on grass. But we never know who are going to play in the final, and how the finalists are going to qualify. Thus, especially locally, we have to accept very radical changes, tactics, even results. Using this metaphor, the basic philosophy of MODEM-project could be described as a *local constructivism* that allows the child to make very radical and individual, even naive and context-oriented mental models in local situations on the way towards more acceptable social and cultural models, concepts. Thus, on the day of the final there is a weak constructivism that is worth attaining, systematically.

2. The idea of "systematic constructivism"

One of the basic questions of mathematics learning, even within constructivism, is the relationship between conceptual and procedural knowledge. The order of the different kinds of concepts has to be planned carefully with recard to logic as well as to learning psychology and history of mathematics. Thus, the first framework of the "systematisation" consists of breaking down mathematical knowledge into concepts, theorems, and procedures (Fig. 2). We try to give the pupil a sufficient number of opportunities to construct the concept attributes and the procedural knowledge based on them. In order to complement this we have to be ready for a long process of concept building in which we separate the phases of orientation, definition, identification, production and reinforcement. This is the second element of systematisation (Fig. 2 and Fig. 3). After this process a familiar problem, which has been used for concept orientation, should be purely an application task. It seems reasonable to assume that especially of basic concepts different children can develop attributes that have a certain one-to-one correspondence.

Because in constructivism the pupil is considered to be a subject who is actively acquiring, processing and retaining information, it requires us to see acquiring knowledge as a dynamic process which is carried out in a situation that is meaningful for the pupil. Contextual problems which are interpretable through the mental models of the pupil can be used for *concept orientation*. By finding the attributes of a concept the pupil can fix the relevant determinants of the concept. This second phase of concept building is called *concept definition* (Fig. 2 and Fig. 3). In the phases of orientation and definition, creative heuristic thinking and productive working by the pupil is needed. In problem solving and mathematics learning, the ability to classify and model situations, and formulate verbal, symbolic and pictorial definitions is probably the most important goal.

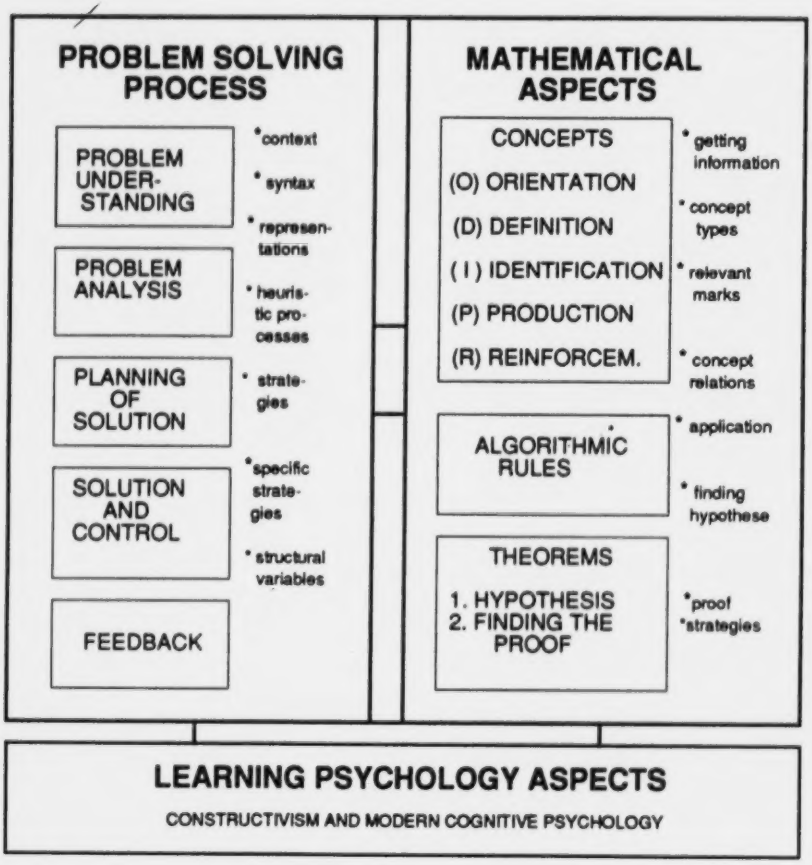

FIGURE 2. *Framework of systematic constructivism*

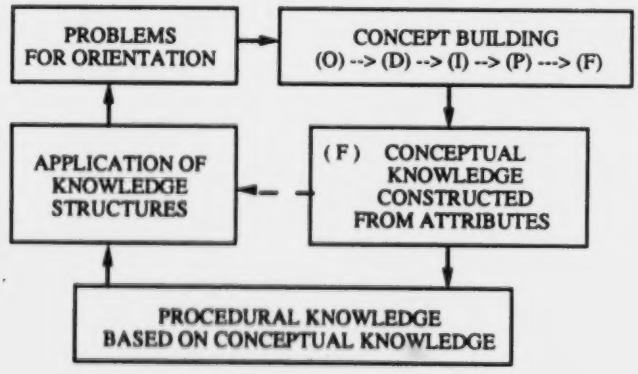
PROBLEMS FOR ORIENTATION
CONCEPT BUILDING
(O) --> (D) --> (I) --> (P) ---> (F)
APPLICATION OF KNOWLEDGE STRUCTURES
(F) CONCEPTUAL KNOWLEDGE CONSTRUCTED FROM ATTRIBUTES
PROCEDURAL KNOWLEDGE BASED ON CONCEPTUAL KNOWLEDGE

FIGURE 3. *Construction of mathematical structures within "systematic constructivism"*

In order to see the concept attributes to be developed as a new, exact and fixed mental model of the pupil, it is appropriate to separate the phases of *identification, production,* and *reinforcement* (Fig. 2 and Fig 3). In the phase of identification the pupil must identify the concept attributes in verbal (V), symbolic (S) and graphic (G) forms. Six kinds of identification task (I) should also be available: IVV, IVG, IVS, IGG, IGS and ISS. Each of the modules begins with tasks that require distinguishing between two elements (Fig. 4) and ends with identification of several elements (Fig. 5). The examples in Fig. 4-7 are from computer display of the learning program LINE explained in section 4. 2.

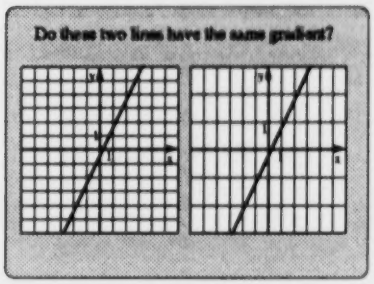

FIGURE 4. *An example of simple identification IGG*

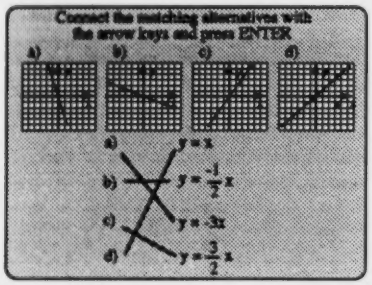

FIGURE 5. *An example of more complicated identification IGS*

By concept production we mean that the pupil has to produce from a given presentation of the concept another representation in a different form. Fig. 6 shows an example of a production from graphic to graphic form.

The development of production (P) requires nine combinations: PVV, PGV, PSV, PVG, PGG, PSG, PVS, PGS and PSS. The tasks of identification and production must be achievable without any complicated processing of information on the student's part.

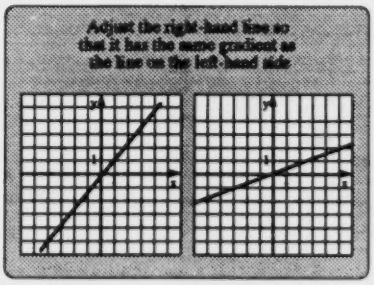

FIGURE 6. *An example of concept production PGG*

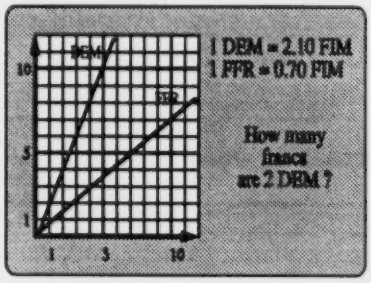

FIGURE 7. *An example of a reinforcement task*

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In the reinforcement phase we want to be free of the role of developing identification or production. All tasks that help automatize the concept attributes and those that use them in problem solving and applications can be trained (Fig. 7). A special class of reinforcement tasks is formed by the tasks for algorithmic rules that can be constructed from the concept attributes.

3. Empirical results

The empirical part of the project consists of subprojects which investigate the learning of certain basic concepts of school mathematics within systematic constructivism. These concepts have been chosen with a view to maximising the application of the results in curriculum development.

In the MODEM 1 study (Haapasalo 1991) one of the most interesting concepts of school mathematics was investigated, namely the concept of the gradient of a straight line. It is one of the most important and powerful examples of representing mathematical information in different forms (verbal, graphic, symbolic; see Fig. 8). It is also a useful and necessary tool for making concrete such abstract concepts as proportionality or equations of the first degree. It helps us to understand the basic ideas of analytic geometry by manipulating simple figures and relationships of points. From the starting point of visual thinking the pupil can more easily reach the higher level abilities of transfigural, analytic thinking. We also recognize the importance of linear functions in interpreting many phenomena or solving problems in different areas, or in using mathematics in everyday life.

The conventional method of structuring school mathematics and the behaviouristic planning and interpreting of "teaching" cannot meet the requirements of systematic constructivism. The verbal, symbolic and graphic attributes of the concept "gradient of a straight line" contain a lot of school mathematics: proportionality, equation, linear dependence, coordinate system, and so on. These are mainly dealt with arithmetically and separated into several different school grade courses. This actually removes the links between the relevant attributes of the same concept "gradient". In our systematic constructivism all these concept attributes of "gradient" were learned simultaneously (Fig. 8).

The empirical study MODEM 2 (Haapasalo 1992) investigated how pupils in the fourth grade learned fractions. The MODEM group learned fractions within "systematic constructivism", and the comparison group was taught in the conventional way from textbooks. One year later the same MODEM group learned decimals within the same kind of learning environments and was investigated in subproject MODEM 3 (Haapasalo 1993). Fractions and decimals represent the most important concepts in the lower stage of the comprehensive school and the motivation for choosing them as an object for

research can be argued on the same basis as the concept gradient in the upper stage. Because textbooks could not be used for learning processes, it was necessary to devise and develop a varied package of learning materials, including computer-based units, all of which are now available (e.g. Haapasalo 1989).

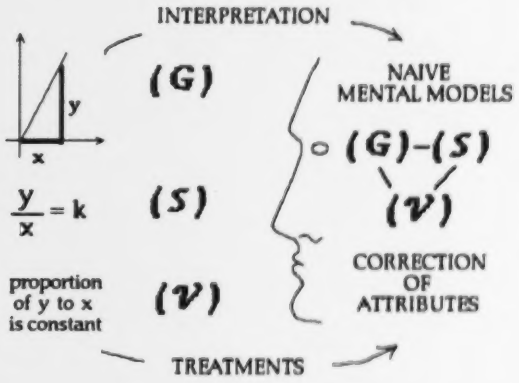

FIGURE 8. *Constructing of attributes of the concept "gradient"*

In all these investigations the cognitive test was planned so as to measure all six combinations of identification, all nine combinations of production, and the ability to use concept attributes in simple kinds of applications. The investigations were planned to mainly study the importance of different phases of concept building and the role of different forms of representation. Observation during the learning period and video tapes provided good opportunities to analyse pupils' thinking processes and working habits in this new kind of learning environment.

It was hoped that the investigation would give answers to the following questions:

(i) At which level of concept building (Fig. 3) do pupils understand the concept?
(ii) What is the role of each subphase of concept building?
(iii) What kinds of tasks measure concept understanding most reliably?
(iv) What kind of information processing is difficult or easy for pupils?
(v) How do the cognitive results of the MODEM - and comparison group differ?
(vi) Do the results of boys and girls, classes and schools differ from each other?
(vii) What kinds of correlations exist between the different kinds of tasks?
(viii) What kinds of erroneous mental models do the pupils have of concepts?
(ix) What is the connection between pupil's result and his/her mark in mathematics?
(x) Are the pupils and teachers ready to use a new kind of learning material?

4. Discussion of MODEM results

4.1. General results

In this section we consider results that are almost identical in all three subprojects.

All investigations confirmed the fact that the process of mathematical concept building is clearly for most of the pupils a long process in which the phases of orientation, definition, identification, production and reinforcement are systematically involved. These phases also form a good structure for cognitive evaluation. The identification phase plays a central role in concept building, and seems to give the pupils the best and most pleasant learning environment for the forming of concept attributes represented in graphic, symbolic, verbal and manipulative form.

Concept understanding was measured most reliably by the production tasks. In most cases the mechanical tasks and simple application tasks are unreliable in this sense. The production tasks including verbal form, especially the verbal-verbal task, are most reliable but also difficult. The identification task produced similar results. The highest internal correlations between tasks are in the production test. Especially the verbal-verbal task also correlates quite systematically with other tasks in the test.

The pupils had poor results especially in the production but also in the identification tasks, and surprisingly performed better in simple application tasks. This might be explained by the pupils' ability to use heuristic strategies, because their mental models of concept attributes were still weak and so influence their thinking. This can be seen clearly from Fig. 10 in section 6.

In the investigation of the effect of information representation (verbal, graphic, symbolic), the thinking processes of the pupils were found to be very different. Nevertheless, the tasks containing verbal representation were systematically difficult for the pupils, and these tasks were found to measure concept understanding very reliably.The name of the concept is one of the details that seems to play a significant role in forming the attributes of a new concept. A pedagogically interesting result is that a horizontal production within the same representation form (see Fig. 6) was significantly more difficult than a vertical production between different kinds of representation. The same result did not hold for identification.

The analysis of conventional textbooks and teaching showed that they focus on the surface properties of the concept, especially on very stereotyped mechanical tasks in symbolic form. An analysis of the pupils´ answers revealed that the pupils developed highly inadequate mental models in conventional teaching, even in respect of the basic attributes of the concept.

Pupils who learned the concepts in constructivist learning environments showed to a very significant extent better concept understanding than the

comparison group, which was taught in a conventional way from textbooks. Their results were significantly better in almost every phase of concept building, and in the majority of the tasks in the test.

The differences in the cognitive performance of boys and girls were smaller in the MODEM-group than in the comparison group, showing that the constructivist learning environments had a more positive effect on the boys' learning. Pupils' opinions about constructivist learning environments were positive. Especially the teachers of grades four and five confirmed the fact that perseverance and consistency in pupils' work also clearly improved also in other subjects.

We shall now examine each subproject and consider certain results that are pedagogigally interesting.

4.2. *Discussion of MODEM 1 results*

In the investigation of the concept "gradient" the computer program LINE (Haapasalo 1989) played a central role. In this learning program each of the five phases of concept building is programmed as an independent submodule and can also be used for computer-based learning as a stand alone program (see Fig. 4-7). However, the main menu allows one to choose any module or an alternative in which the program automatically takes care of all five modules. It means that concept building and the pupil's attempts are controlled and evaluated by the computer alone. The order of the modules, as well as the order of the tasks in each module, can be arbitrarily changed by the teacher before starting the program. This makes it possible to carry out a continuous study of the meaning of each subphase of concept building.

The pupils in eighth grade (N=286) learned the new concept of "gradient" in two hours in a learning environment managed by the computer alone. For each of three subgroups the phase of identification, production or orientation was skipped in the learning program. The pupils (referred to as the CBL - group) immediately answered questions about the new kind of learning environment, and after three weeks they took a cognitive test. The comparison group learned "gradient" with the same tasks in the same learning environment, but without the computer.

This investigation showed most clearly that the identification phase plays a central role in concept building. It seems to give the pupil the best and most pleasant learning environment for the forming of concept attributes (see Fig. 4-5). Skipping the identification phase caused a lot of difficulties in concept building which could be immediately observed during the learning time. Such pupils' cognitive results were significantly poorer and their opinions of the learning significantly more negative than those who learned with identification. Skipping the orientation phase did not cause as dramatic changes but it had

some effect of the same kind. However, skipping the production did not cause any statistically significant differences in the cognitive test, but had a positive effect on the pupils' opinions of the learning! This is an interesting result, because textbooks consist mainly of production tasks, which are very limited in type.

There were some interesting differences in the opinions of boys and girls on computer-based learning. The girls had clearly less experience with computers but also more fears of computer-based learning than the boys did. These fears were found to be fears of the program more than fears of the computer as a technical instrument. The girls found learning with computer more fun, interesting and exciting than the boys did. Neither the pupils' marks in mathematics nor their familiarity with computers had any correlation with their opinions of the learning environment or computer-based learning.

During observation of normal eighth-grade school teaching of the concept "gradient", it was noticed that most activity was concerned with the training of "surface properties" of the concept, and with very standardized behaviours. Very interesting results were seen when comparing the performance of the CBL-group with that of comparison group (9th grade). To the latter the same concept had been taught during two school years in a conventional way within different courses. In the same test the differences between CBL- and the comparison groups were surprisingly small as the average points in Fig. 9 show.

The analysis of pupils' thinking processes during CBL-resulted in a recognition of the fact that the pupils have extremely erroneous mental models of many of the basic concepts in school mathematics, even of the concept of a straight line. Especially when the computer is involved, it seems possible to organize new kinds of learning environments and to develop pupils' working habits. Nevertheless, a comparison of the results of the group that learned without the computer with those of the CBL-group yielded no conclusive evidence that learning with the computer has any positive or negative effect on the cognitive performance of the pupils, or on their opinions of this kind of constructivist learning environment.

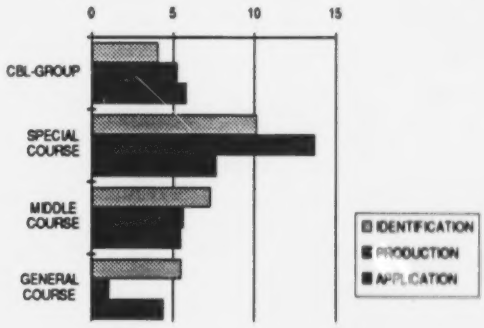

FIGURE 9. *Cognitive performances of each group in MODEM 1 (comparison group is distinguished by the quality and quantity of the course)*

4.3. Discussion of MODEM 2 results

The empirical subproject MODEM 2 investigated how pupils in the fourth grade learned fractions. Pupils in the MODEM -group (N=55), who learned fractions in constructivist learning environments, showed to a very significant extent better concept understanding than the comparison group (N=94). Their results were significantly better in every phase of concept building, and in 80 per cent of all tasks in the test.

The differences in the cognitive performance of boys and girls were smaller in the MODEM-group than in the comparison group, showing that the constructivist learning environments had a more positive effect on the boys' learning.

Pupils' marks in mathematics showed a significant correlation with their performance in all subtests excluding mechanical tasks. These tasks also had a poor correlation with each other and with other tasks. Every application task in the middle of the test had a corresponding mechanical task at the end of the test reguiring the same arithmetic operation. Fig 10 shows that, in spite of the large number of mechanical tasks in textbooks, the performance of the comparison group on such tasks was poorer than the performance of the MODEM-group, which did not have any drilling at all. Nevertheless, the MODEM-group learned percentages into the bargain. The explanation is probably to be found in a deep and comprehensive formation of the concept, something which systematic constructivism can provide a solid basis for, as shown in Fig. 3.

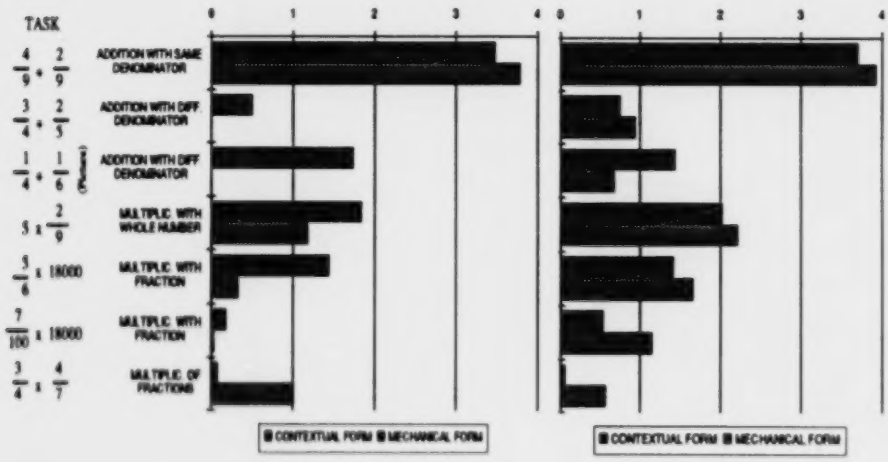

FIGURE 10. *Pupils' cognitive performance in mechanical and corresponding context - oriented verbal tasks with f ractions. (MODEM -group on the right and comparison group on the left).*

In the comparison group finding a suitable arithmetic shema was dependent onthe contextuality of the task. This can be shown most clearly in the task requiring the addition of fractions with different denominators. Surprisingly, the MODEM -group performed mechanical tasks even better than context oriented-tasks. The chaotic variety of the results of the comparison group shows that textbooks do not help the learning of concepts, not even the learning of mechanical routines. It also shows that the results of those investigations in which concept understanding has been measured only with mechanical tasks must be interpreted critically.

The learning experiment also provided good opportunities to observe pupils' behaviour in problem situations. Most often pupils interrupted the problem solving process when trying to find or carry out the idea of the solution. If this was the case, the pupil usually asked for help from another pupil or from the teacher. One of the most important pedagogical goals was to develop pupils' ability to communicate and think together without the teacher's help. In the orientation phase pupils were highly motivated and had no difficulties in understanding the problem or thinking up ideas for solving strategies. Only in a few cases did the pupil become frustrated and totally interrupt the process.

Because the plan was to use many kinds of working methods based on pupils' spontaneous activity in the learning processes, "teaching by demonstration or model" was actually not allowed. Nevertheless, teachers still asked the pupils too many questions. The time used by this kind of teaching method was about 40 per cent. In spite of inservice training before the learning period, teachers had difficulties in gaining a deep understanding of the principles of constructivism, and therefore occasionally one could even observe very strongly behaviouristic moments. This probably had some negative effect on concept building as a whole. The teachers said that they were automatically so highly motivated to "teach" that they had to fight against this feeling all the time. In the beginning pupils made angry remarks like "teacher, why don't you *teach* this to me?". After a few weeks one could hear comments like "why can't we *think this out*!"

4. 4. Discussion of MODEM 3 results

In the fifth grade the same MODEM-group that had been involved in MODEM 2 one year earlier learned decimals within systematic constructivism. Pupils were initially oriented to the task by using several kinds of digital and analogue measuring instruments. The focus was to deal with decimals together with measurement and units and thus to give the children an opportunity of receiving a correct mental model of accuracy from the outset. To this end the children measured the same object with many instruments, with sticks, meters,

decimeters, centimeters, millimeters etc. Using the digital thermometer and voltage meter it was possible to change the accuracy read from the display. With a display of 1.4 volts, for example, one could increase the accuracy and get 1.36 or 1.42 volts. This showed the basic fact: you can never get more information than the numbers show! This was very natural for the children, because they think concretely. The textbook routines for adding zeros to decimals are not only incorrect but the most effective way to make the concepts of measurement and accuracy difficult for the children.

Because the MODEM-group consisted of three classes it was possible to study the meaning of the orientation base. During the first lesson one class had a number box with the names "tenths, hundreths, and thousandths" on the blackboard. In addition to this, the second class also had the names "decis, centis, and millis" on the sketch. The third class had to describe, explain, write, etc. the results of measurement without any orientation help on the blackboard. Surprisingly, the second class liked to use the "unit terminology" for most of the discussions. This group actually learned to change units, into the bargain! This has immediate implications for the curriculum such as handling per cent procedures as a natural part of the phase of concept reinforcement (see Fig. 3).

As far as the investigation results are concerned, decimals seem to be much easier for the children than fractions. In the cognitive test the children, in fact, performed more consistently in all subtests. The correlations between context-oriented tasks and the corresponding mechanical tasks were essentially higher than in the case of fractions.

The MODEM-group not only had a significantly higher level of performance than the comparison group but also better results than the pupils in the upper comprehensive school or secondary high school, especially in tasks containing verbal representations. After conventional teaching even a pupil in the last-mentioned group seems to have wildly erroneous mental models of units and accuracy.

5. Some implications of the results for the curriculum

All these three concepts turned out to be difficult for the pupils, because they did not attain the higher levels of concept understanding. It seems that learning within systematic constructivism could lead to significantly better results, especially because the pupils seem to like this kind of learning environment.

In order to change teaching habits it is necessary to make new kinds of learning material and organize effective in-service training for teachers. Observation of lessons showed clearly that the behaviouristic teaching tradition from textbooks is very strong. The results of MODEM -groups prove that the strong symbolic, mainly mechanical mathematics in textbooks should be decreased and an attempt should be made to give pupils better opportunities

for mathematical thinking. The learning of concepts should be started, not from arithmetical or algebraic, but from context-oriented situations. For the formation of deep and comprehensive understanding of concepts it is probably necessary to define more clearly certain focus areas in different school grades. For fractions, for example, the forming of mathematical structures within Fig. 3 took several months. This gives rise to the following questions: Is it realistic to assume that teachers can move the focus from the planning of lessons to the planning of constructing processes that last weeks, months, maybe years? How should the curriculum be formulated as process goals which include these new modes of pedagogical thinking?

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Social Constructivism and Assessment

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Åbo Akademi

The function of assessment within a constructivist - and particularly a social constructivist - framework for mathematics and science education is analyzed. The special role of the teacher is emphasized. Some suggested foci for assessment are presented, as well as a selection of methods of assessment, compatible with a constructivist view.

1. Constructivism and assessment

With the present popularity of constructivism as a guideline for teaching and learning mathematics and science, the problem of assessment has become increasingly evident. Constructivism holds that knowledge is something that is actively constructed by the student - it is not passively received from the environment. The process is adaptive and organizes the experiences of the student. According to that view, transmission of knowledge from one person to another is an impossibility or a contradiction in terms.

The world of knowledge of one person is not directly accessible by another. How then, is it possible to assess the quality of another person's knowledge in an objective way? Assessment would seem to require some kind of comparison of the knowledge of a student with a standard, but the comparison itself requires communication which is expected to be incomplete. The best one can hope for is a general picture which is put together from pieces of information.

There is also the problem of the standard. Is it enough to consider mathematics or science as resting on objective truths that can be used as knowledge standards? The discoveries of mankind are based on this kind of assumption - that there exists something objectively true and real. In the case of mathematics - that there exists true mathematics waiting to be discovered. For the students, then, assessment criteria can be derived from what is known about true mathematics.

The weak point in this is, from the view of constructivism, that it puts shackles on the process through which knowledge is constructed. In fact, it means that students construct two kinds of knowledge, real knowledge which agrees with real mathematics or real science, and other knowledge of a personal type. Real knowledge is then actually discovered rather than constructed, and the

pseudoindependent process in the classroom is one of guided discovery. There would seem to be no need to rename that process "constructivism".

Radical constructivists reject the possibility of obtaining real knowledge about the world, including mathematics (von Glasersfeld 1987). The construction of knowledge is not guided by truth criteria. During the process of construction, students experience various cognitive conflicts -viable knowledge is knowledge that comes out of these conflicts and gives, at least temporarily, an understanding of the world of experience. The sense of a direction in the development of knowledge is spurious. The process is rather similar to the one introduced by Darwin - the knowledge that is most adaptable to a variety of situations is the knowledge that survives.

Clearly, assessment in mathematics education is not easy to handle if you profess to be a radical constructivist.

The concept of viability may be attractive to others than radical constructivists. It may be combined with the view that there actually exists a possibility of defining a direction of development of knowledge. It is characterized by a sequence of improved models for the structure of the real world, an idea which is acceptable to "weak" or "trivial" constructivism. In such a case it is more reasonable to derive assessment criteria from what is known about the real world.

For any constructivist, there remains the problem of formulating criteria to be used in formal assessment, acceptable to society. In what way can you compare students with each other with respect to their proficiency in mathematics or science without resorting behavioral objectives? The problem has become much more pressing with the politically initiated movements towards detailed curricula and accountability in the teaching profession, evident now in many countries.

2. Social constructivism

Despite the differences in views between various kinds of constructivists, they will agree that the construction of knowledge takes place with social interaction as a major influence upon the individual. Students are affected by their families, teachers, and other students, and they learn from their total experience. To further emphasize the importance of these social connections, the term social constructivism has been suggested independently by several authors. Ernest (1991) traces the concept to the view of the development of science in society represented by Lakatos (1978) and Popper (1979). He describes (1990) his own theory of mathematics in society as inspired by Bloor (1976) and Restivo (1988). Björkqvist's

view of science education (1990) similarly was inspired by Solomon (1987). Others that must be mentioned are Weinberg & Gavelek (1987) and Lerman (1989).

An important element of social constructivism is consideration of the knowledge that is common to a specific group of people. The mathematical or scientific knowledge that dominates in a specific culture may be called collective knowledge, and our special interest is in the relationship between the collective knowledge and the knowledge of the individual.

If your view is that of a radical constructivist, this collective knowledge is the best that you can get as a standard to be used in assessment. Ernest (1991), following Popper, calls it "objective knowledge", using a term that might also be associated with properties of the real world. Calling it "collective knowledge" avoids confusion - but in either case it refers to Popper's World Three.

If you are not a radical constructivist, you may choose to view collective knowledge as a set of models corresponding to or converging towards properties of the real world.

Collective knowledge is characterized by its ties to a specific culture or group of people at a specific moment in time. Mathematics and science are viewed as social constructions. New mathematics and new science is created by individuals. It is included in collective knowledge through a process of acceptance following criteria that are socially defined, and the same kind of process may lead to rejection of previously accepted pieces of collective knowledge. It is important to note that falsification is only part of this process. The history of mathematics and science shows numerous examples where cultural conditions in general or personal connections have been decisive for the development of new scientific knowledge. What we call mathematics and science is a collection of concepts and methods that have proved viable in society.

Collective knowledge may be viable to a certain group of people while at the same time it is not viable to another. Examples of such knowledge are to be found in the voodoo "culture", astrology, and the use of the abacus. When different cultures collide, there is a confrontation of views that may lead to assimilation, modification, or disappearance. Ethnomathematics (Lave 1988) has clearly shown that the process need not lead to homogenization of the collective knowledge, since the process often is influenced by competing criteria of viability.

The other aspect of the interrelationship between collective knowledge and the knowledge of an individual is the way in which the individual includes pieces of collective knowledge in his own construction of knowledge. Social constructivism emphasizes the unique capacity of man to receive abstract information through oral or written communication. It is a foundation for cultural traditions and for the survival of scientific knowledge from one generation to another. The system of

organized education itself expresses the importance of learning through social interaction.

When a student in school constructs his mathematics or science he will use his teacher as a reference. His assessment of the quality of his own ideas will be a process of a predominantly social kind. Acceptance by the teacher will be an indication that he may continue to reason in a similar way in the future. There is rather less self-assessment using mathematical or scientific arguments. There may, however, remain a considerable amount of individual knowledge which may differ from collective knowledge and nevertheless be viable for the individual. This is well-known from studies of personal conceptions (Pfundt & Duit 1991).

A description of the mechanisms that make an individual change his conceptions cannot be given using just parallels with the development of mathematics and science in society. Individual learning belongs to the domain of psychology, and requires a (constructivistic) learning theory. Self-assessment would seem to be an important component in such a theory. If there exist personal criteria for the quality of knowledge, they may be inaccessible to others, and of course they can be expected to be changing as a result of social interaction.

The concept of viability at the individual level is thus subjective. There are related concepts that appear reasonable to associate with the existence of viable conceptions, above all the subjective feeling of understanding (Björkqvist 1990). On the other hand, there is considerable documentation (e.g. Meyerson & McGinty 1978) of behavioral patterns that are stable, giving evidence of individually viable mathematics, while at the same time there is no evidence of understanding, other than in an instrumental sense.

The latter can be seen as an exemplification of pragmatism at the individual level. The social context very often is such as to present mathematics as a useful subject, a tool. And of course, there is reason to see it that way. Viewed from the point of future success for the individual in the struggle for survival, an investment in mathematical knowledge opens up many possibilities, and mathematics usually is just a means towards other ends. Self-assessment and formal assessment of mathematical knowledge must acknowledge that.

However, as social constructivism also emphasizes, mathematics and science are dynamic and respond to changes in society. Their usefulness at the social and cultural level presupposes a continuous influx of new ideas, and those must come from individuals. A restriction in formal assessment, focussing only on mathematics and science that has previously proved useful, will quench that flow.

Man has the unique capability of investing in future knowledge which will help him as a species in a variety of ways. Such mathematical and scientific knowledge may not be recognizable at the time of its birth, but the methods of

assessment should include mechanisms that encourage creativity. The teacher is in a critical position in this difficult task.

A necessary property for a conception to be socially viable is that it can be communicated. At an elementary level this requires sufficient agreement regarding concepts and methods employed by two or more people at a particular moment. In mathematics and science the definitions of concepts have been made systematically exact and the methods have been standardized, i.e., one tries to maximize communicability by use of conventional concepts and methods. Those are to be counted as part of our scientific culture. Traditional assessment is directed towards them to a considerable extent.

In a constructivistic perspective, too, there is reason to assess the relatively static aspects of the mathematics and science of the students. The aim is not to assess reproductive ability, but to promote communicability and interaction with the collective mathematical and scientific knowledge, with all the social expectations that go along with it.

Assessment influenced by constructivism also includes dynamic aspects, assessing the process through which the student creates knowledge that is new to him and assessing the quality of the constructs with respect to viability in foreseeable and unforeseeable situations. It entails increased attention to the student's present view of mathematics and science. It also requires assessment of his active endeavors to change his conceptual world quantitatively and qualitatively, independently and in collaboration with other individuals in society. It furthermore involves assessment of the degree to which the personal knowledge is context-dependent.

Assessment from a constructivist perspective does not require old methods of assessment to be discarded, but rather that they be complemented with new methods. It is imperative that such new methods are rendered acceptable in society, since the formal way of assessment normally influences the way in which education is organized.

> You get what you assess.
> You do not get what you do not assess.
> [Resnick & Resnick 1991, 59]

Social constructivism implies a strong emphasis on the role of the teacher for the process by which the students construct their knowledge. The students test their conceptions primarily on the teacher and gradually, with the support of the teacher, independently by self-assessment. The teacher analyzes their way of thinking and attempts to predict the effects when it is confronted with other ways of thinking that may predominate in society. Part of the teacher role is promoting

the development of metacognition, making the student aware of his own thinking and that it may differ from that of others.

The teacher is the representative selected by society to keep the collective knowledge - the culture of mathematics and science - available to the students. Considering the changing nature of collective knowledge and the conditions of society, this is not a definite, but a provisional position of authority - a matter of confidence in a professional. The picture includes the expectation that the teacher is capable of placing the conceptual world of a student in a developmental perspective better than anybody else. The teacher also is the one to make decisions in cases where different criteria for the viability of knowledge are to be balanced against each other, for instance individual creativity and precision in conventional terminology.

Social constructivism assumes that the teacher has visions of the social development of science and mathematics and that those visions are actually put into use in the planning of educational activities and in the assessment of the outcomes of them. At the same time it is important that the teacher does not see physics, for example, as an isolated system. The concepts of physics are loaded with associations that reflect concepts, processes and values from all sectors of society.

It is remarkable that social constructivism, while firmly based on the view that each individual constructs his own knowledge, should imply such an important role for the teacher in that process. As a theory, it appears to restore to the mathematics and science teachers much of the status that they formerly had. The expectations with respect to teacher knowledge and skills are quite different, however.

3. Foci for assessment

Since the social constructivist view makes it reasonable to consider educational goals for a society, it is possible to define focal points for assessment in agreement with that theory. An attempt to do so is made below, with no particular order of importance or relative weight.

Mathematics and science are both social constructions to which each individual contributes in his specific environment. Activity, rather than passivity, is something that should be promoted. One focus for assessment should thus be *activity*.

Students should gradually become aware of the various roles of mathematics and science in society. Clarifying the role they play in one's own life is a step in this direction. A second focus for assessment then is the general picture of

mathematical and scientific activity that each student has, his *view of mathematics and science*.

Since the basis for individual construction of knowledge is experience in a social environment, e.g. receiving information from others and testing your own ideas against the ideas of others, a third focus for assessment is *social interaction*.

The ability to interact with other human beings requires a large enough overlap of personal concepts and terms. Mathematical enculturation promotes this. It is essential that one acquires an ability to give verbal and written messages with mathematical and scientific content. One further focus for assessment then is *communication*.

A unique feature of mathematics and the exact sciences is the possibility of forming abstract concepts and successions of abstractions. They are valuable in the communication of knowledge, and the capability of handling abstract concepts is a central goal in education. Assessment should therefore give attention to *concept formation*.

Concept formation starts with concrete experiences that can be made abstract and idealized. Variation of the contexts of the experiences promotes viability of the concepts. Testing one's own ideas in a wide variety of applications similarly enhances the usefulness of abstract concepts. There is great reason to focus assessment on *applications*.

This is in line with the view that a principal role of mathematics in society is as a support for other human activities. Traditionally, mathematics also involves precision, and the need for people capable of performing mathematical calculations swiftly and correctly has continuously been great. It is evident that assessment in mathematics should be directed towards *correctness*, that is, agreement with collective mathematical knowledge. On the other hand, the necessary flow of new ideas into mathematics and science requires attention to evidence of *creativity*.

Mathematics is characterized by its use of reasoning and formal logic. Social constructivism would imply mathematical logic to be a social construction. At the individual level, it evolves out of contextual variation in experiences related to mathematics, and social interaction plays an important role. Non-mathematical logic will typically be present, and one cannot presuppose reasoning according to fixed logical principles. The assessment of *reasoning and logic* thus should reflect the

interrelationship between collective and personal mathematics. More generally, the fact that concepts and methods are personal, although they share a common cultural background, should gradually become evident to students. To varying degrees, and in specific contexts, students are aware that their thinking may differ from that of their fellow students. A possible focus for assessment thus is *metacognition* when conditions are favorable.

As a matter of principle, assessment should be aligned with the educational methods in use. The social constructivist view allows for a wide variation in methodical emphasis - mathematics and science education must be dynamic and respond to changes of conditions in society. The long-term goals cannot be reached through the measures taken by single teachers, but with large enough agreement concerning what may be beneficial for the individual and for society, there is good hope for a synergy effect. In that case the variation in educational methods may provide the creative innovation that a stable but non-stagnant educational culture needs.

4. Methods of assessment.

Beside traditional methods of assessment, which may be employed in a natural way from a social constructivist perspective, there nowadays exist both new methods and warmed-over methods adapted to the ideas of today. A list of such methods can only be exemplifying, and primarily serves as inspiration for a teacher who wants to develop a personal set of methods of assessment. Some methods to consider are the following.

Diary of interaction with students

Comments concerning details in the learning process of individual students are entered in a diary, preferably with some regularity. Other entries record actions taken to promote learning (Clarke 1992).

Student portfolios

In accordance with practice within education in arts, e.g., work by a student during a certain time period is gathered in a portfolio. If those pieces of work are chosen in a representative way, they give a qualitative description of the development of

the student. For more information about assessment using student portfolios see Stenmark (1991).

Concept maps

Concept maps (Novak & Gowin 1984) describe the students' personal concepts in science and relationships between those concepts. Contextual and affective relationships may be included. The maps can be considered as concrete representations of mental schemas. When using them for assessment purposes (Gibson 1991) one may study the number of links between concepts, the existence of isolated concepts and key concepts, and the way the student enters new concepts in his concept map.

Self-assessment

The student and the teacher may come to an agreement about a short written report to be given by the student describing his own relationship to mathematics, physics, etc. (Clarke 1992). Even if it is not anonymous, it should be confidential. The report may include, e.g. personal ambitions, and it therefore gives a possibility to assess how realistic the student is about himself. The report may very well be repeated with some kind of regularity.

Phenomenography

Phenomenographic interviews (Marton & Svensson 1978), i.e. deep-going studies of the conceptions students have of a certain phenomenon, offer a chance to obtain a picture of the variation of those conceptions within a class. The conceptions are categorized and a qualitative analysis of the categories is made.

Test items constructed by the students themselves

Upon agreement between the teacher and the students, and applying a system that is as fair as possible, test items relating to a certain course section are constructed by the students for actual use. These items make it possible to assess the overall views of the section and to identify those concepts that are central for the students. At the same time they give information about the student's own assessment of his knowledge and that of his fellow students.

Test items that require decisions about which data are to be used

Situations with an excess of available data give the teacher a possibility of assessing constructive activity by the students, particularly if it requires a certain effort to obtain the necessary information. If several models are possible with one set of data, the ability to choose an appropriate model will be assessed.

Tests containing solved exercises

Completed solutions may contain errors that are chosen to alert students about critical stages in the solution. The students may be asked to find the errors, to describe their nature, and to correct them. In other cases two correct solutions, differing in structure, may be presented with a request that they be compared qualitatively.

Practical tests

Applications of mathematics (Clarke 1992) and physics occur in a natural way when concrete objects are manufactured. It may involve, e.g. measurement and determination of the conditions for equilibrium. Relatively simple such exercises can be included in practical tests. Test items taken from everyday life offer a chance to assess the context-dependence of learning and to relate school knowledge to other kinds of human activity.

Extended tasks

A two- or three-week home assignment on a single theme, a special kind of research, is a method of assessment that makes great demands. It may require a written report on background, methods, and results. It is expected that the student should create knowledge that is new to him and to structure it so that it can be communicated. Assessment of this type has been included as part of a matriculation examination (Anon. 1990).

It should be added that none of these methods of assessment is intended as an end in itself. Assessment presupposes that it is followed by action - so also assessment that is inspired by constructivism.

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Can Finnish Elementary School Pupils Think Logically?

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The article analyses the possibilities of deductive reasoning from the perspective of cognitive psychology. The background is provided by Johnson-Laird & Byrne's book 'Deduction', which brings together studies of deductive reasoning and develops the mental model theory. The present study examines the inference models of 40 pupils (aged 9-13) using 26 tasks in an interview. The weakest reasoners (approximately one fifth) did not have a clear model of thinking. Most pupils were uncertain in their inferences and occasionally formed situation-specific models of thinking, using causal or analogical explanations. Only about a quarter of the pupils made logical inferences in the majority of situations. The results largely support the mental model theory, but it does not give a fully picture of the pupils' thought processes. The pupils' deductive reasoning also fairly generally involves content-specific thought, which is also acceptable in teaching on constructivist lines.

1. Deductive reasoning as part of cognitive thought

Inferences related to thinking activity are often classified as follows: deduction, induction, analogical reasoning, creative thought and various combinations of these. Deduction is logically binding in that conclusions are reached on the basis of given premises according to certain inferential rules. I shall confine myself here to an examination of deduction related to cognitive processing where the inference must be valid, i.e. true inferences are derived from true premises.

Deductive reasoning is not performed merely according to the rules of logic. In human thought various explanations are created for the inferences. The explanation can also be logical but in most cases the inference is accounted for by a causal or final explanation. In a causal explanation a cause-and-effect relationship connected to a real-world phenomenon is presented. The basis is some general law that is interpreted for this particular case. In a final explanation the event is described by means of its purpose. In this the performer, the action and its purpose are combined into a whole. In addition to deduction, causal and final explanations also contain induction and analogical reasoning, generally all combined together. Since we usually examine other people's thinking on the basis of their explanations without any information about the form of their reasoning, the part played by

deduction in reasoning easily remains unclear.

In making deductive inferences we do not seek merely the trivial consequences of premises but also useful conclusions which are not immediately apparent from the premises. Here, however, we encounter the paradox of inference proposed by logicians (Cohen & Nagel 1966, 173-176): Inferences cannot be both valid and useful. If in an inference the conclusion is not contained in the premise, it cannot be valid; and if the conclusion is not different from the premises, it is useless. For this reason it is easy in human thought to break away from reasoning based on formal logic, or at least to use the premises loosely. It is difficult to draw the line between logical reasoning and other reasoning.

The informational (cognitive) thinking process in humans involves them forming their own mental models. In order to form these they require concepts and the relationships between these concepts, which develop in every person on the basis of their own knowledge. Logical thought is explanation by means of deduction within the framework of this individual mental model. Mastery of logical thought is an important goal in school instruction, which makes deductive reasoning an interesting subject of research in the field of cognitive thought. The following problems will now be preliminarily examined:

a) What kind of reasoning models do Finnish elementary school pupils have?

b) Is there any point in teaching logical thought in schools?

2. Three models of deductive reasoning

Initially I shall present the models of reasoning in the form they are classified in Johnson-Laird & Byrne's summary report 'Deduction' (Johnson-Laird & Byrne 1991):

1) Deductive reasoning based on formal rules of deduction at the computational level. The formal operations used may be presented as algorithms and can be programmed. A typical device for this type of resolution is the programming language PROLOG. The treatment of symbolic structures has advanced to such an extent in the logic of artificial intelligence that the adequacy of complicated inferential processes, for example, is under investigation (e.g. Genesereth & Nilsson 1987).

2) The use of content-specific rules of inference in deduction. These can be partly programmed as expert systems (cf. Johnson-Laird & Byrne 1991, 31-34), but inference often takes place in the form of cognitive construction based on some item of common knowledge to which background suppositions are attached. In the latter case the person himself forms a rule

or norm from common knowledge which is linked to the contents of the working memory, on which basis the inference is made. In addition to deduction, analogical and inductive inferences are involved in the interpretation of common knowledge.

3) Inferences based on the theory of mental models. Here the person forms his own mental models of the situation which have the same structure as the situations which present them. According to the mental model theory the deductive process advances in three stages of development, first by filling out the information from the premises and constructing an internal model, then by making a sketchy description of the constructed model while simultaneously testing it, and finally by investigating the alternative models deriving from the premises through an examination of the validity of the inferences. (Johnson-Laird & Byrne 1991, 35-39).

In Johnson-Laird & Byrne (1991) the rule theory presented in 1) is contrasted with the mental model theory presented in 3). The authors present abundant empirical evidence showing that deductive reasoning in humans can be best interpreted according to the mental model theory. Nevertheless, it is not a guarantee that the inference will be correct since people do not discover all the possible models from the premises. In operating according to the model theory people make use of abbreviated inference rules. Inference making based on content-specific rules, outlined in 2), has been left aside here since content-specific rules require additional inferential machinery for it to be possible to construct the theory in a context-free manner (Johnson-Laird & Byrne 1991, 195). Omitting this intermediate mode may be justified from the point of view of developing the mental model theory, but at the same time we exclude one of the typical forms of human reasoning.

3. The thought processes of pupils

In order to acquire concrete information on the reasoning used by pupils in the Finnish comprehensive school, I arranged an empirical study of pupils (n=40) in the elementary stage of the Jyväskylä University Training School in the spring of 1992. Pupils in grades 3 to 6 (ages 9-13) were interviewed using six different inferencing situations with a total of 26 subtasks (inferences). This research is reported in a Jyväskylä University Teacher Education Department publication (Malinen 1992). As part of the structured interview the pupils' explanations were observed, but it was also possible to explain in part the pupils' thought processes, or, in other words, the ways in which they constructed models for their inferences.

In one task the pupil was given this piece of information: "If I receive a present, I am friendly." The pupil was then asked which of the following statements contains exactly the same idea as the earlier statement: "If I don't receive a present, I am not friendly." or "If I am not friendly, I don't receive a present." The latter statement is equivalent to the initial information (the logical counterposition), and the consequence (I am friendly) is more general than the premise (I receive a present), which means that it should be possible to distinguish the statements as a logical inference.

In another task the initial information was as follows: "Jari is at work or Pasi is fishing, but not both at the same time." Then an additional piece of information was given: "Pasi is not fishing. What can you conclude?" The form of reasoning is affirmative disjunction, which according to Johnson-Laird & Byrne (1991, 52-55) should be more difficult than modus ponens or modus tollens reasoning. The study included all these forms of reasoning as presented in the same situations.

Especially in simple situations pupils reasoned logically directly from the premises according to the mental model theory. Obviously these pupils also used abbreviated thought processes, but there were only random occurrences of pupils' examining alternatives. There was no evidence of reasoning based on formal logic (rule theory). The pupils' commentaries, however, also contained causal explanations, in which case reasoning was no longer clearly logical or in accordance with the structure of model theory. Pupils would then use some general view appropriate to the situation as their basis and make their inferences accordingly. Sometimes new premises were introduced and in this situation applying the principle used by the pupil could lead to an incorreect inference. This corresponded to reasoning based on content-specific rules (the second model).

The first task mentioned above produced only a few correct answers. The thought process here, too, appeared to be random and there was no logical reasoning discernible in the justifications. Approximately half of those answering the second question replied directly that Jari would be at work. The reasons given also accorded with logical thought. Some of the pupils, however, reached their own causal conclusion, for example: "If Jari and Pasi are friends, then Jari wouldn't have gone to work; they would have stayed together." In this case the pupil has formed his inference model analogically on the basis of some familiar situation. Sometimes reasoning of this type is successful; in this case it was erroneous.

Some pupils found it generally difficult to make clear inferences. They made guesses without being able to give reasons or without even properly understanding the task. The sample included 7 pupils of this type who were only capable of correct reasoning in basic, concrete situations. In terms of school success they belonged to the weakest third. Even though the differences between pupils were

clearly evident, they all based their thinking on the same logic. The weakest pupils had poor concept formation and poor mental model formation.

The sample contained approximately equal numbers of boys and girls. There were no essential differences between the sexes in logical reasoning. Similarly there were only minor differences betwen pupils in different grades. Even in grade 6 there were a few pupils whose reasoning was particularly random.

On the basis of this empirical data it is possible to identify in the pupils the following styles of thought process in deductive reasoning situations:

1) The inference is made mostly by guessing or by random attempts. The pupil has not formed an actual model of thinking.

2) The inference is made uncertainly on the basis of the given information. The pupil may give an explanation by forming a content-specific model which permits causal explanations. The use of premises is loose and the formation of models of thinking random. The pupil is dependent on content-specific rules and makes his or her own generalizations and analogical conclusions.

3) The pupil is in control of the situation and reaches conclusions on the basis of the premises. Nevertheless, logical inference rules are not specified, but in some situations the pupil examines alternatives and tests their validity. In this case the pupil is operating in accordance with mental model theory.

The thinking process styles presented above are not actual cognitive styles, but they describe differences in the precision of deductive reasoning. Up to a point, they provide support for the usefulness of the mental model theory; in my opinion, however, the model theory should be supplemented by also describing defective deductive reasoning. In this way the connection between perception and the use of concepts will be made apparent. We are not, however, necessarily dealing with three stages in the development of cognitive thought which should be reflected in teaching, since the use of content-specific rules may be the best style of reasoning in certain situations. Such situations, however, do not make it possible or necessary to use precise deductive reasoning. There is a particular need to have control and use of causal conclusions in many contexts. Even though the mental model theory has shown its applicability in the interpretation of deductive reasoning, pupils in the elementary grades of the Finnish comprehensive school make no use of it except in narrowly-defined situations.

4. Mental model theory and constructive thought in teaching

Pupils are unable to make valid deductive inferences if they have deficient knowledge. The accuracy of their reasoning depends on how well, according to the mental model theory, they progress to the second or third stage of development. The problem is in fact whether we can increase the precision of the pupils' reasoning. With experience the child develops the ability to understand language, similarly the processing capacity of the child's working memory increases. It is also possible to enhance the reflective skill of one's own performance (Johnson-Laird & Byrne 1991, 204-205). The problem does not lie in the lack of logic as such since everybody appears to possess the same inborn logic. One must merely get it to work well with one's own control system.

Precise deductive reasoning skill, as defined in the mental model theory, has been set as a goal in the present school system. It would help pupils to construct their own internal models, to test the validity of their reasoning and to form concept systems. There is, however, considerable variety in pedagogical development. Multiple forms of constructivist thinking are presented in Piaget's name. One aim is to develop the child's thinking styles in the same way as a computer is programmed. This, however, easily leads to the programming of logical processes according to the rule theory, something which in the study of deductive thought has proved to be alien to pupils. It is totally unnecessary to teach children thinking of the artificial intelligence type since it is an extremely inflexible model of reasoning. On the other hand constructive thought is adapted to freer heuristic processing in practical situations where the focus of interest is not on the form of the reasoning as long as the pupil's own modelling proceeds well. Even though deductive reasoning is not present explicitly, the need for logical thought is implicitly included.

To date there has been no direct presentation of experiences in the teaching of reasoning models according to the mental model theory. It is true that this was dealt with in the 1960s in Polya's heuristic reasoning processes. This, too, may develop into a formal use of search algorithms, also experimented with at the same time. Dienes's system in the 1960s (Dienes 1960) sought logical inferences in another way. The ideas of Bruner and Dienes may well be the clearest starting point for constructivism in mathematics teaching. Particularly Dienes's use of logical blocks and the nature of the problems set for pupils came very close to the teaching of a mode of thinking based on model theory. The idea was good, but its application has apparently remained a formal exercise without any connection to the formation of mathematical concept structures in pupils or to practical application.

I also question the aim of endeavouring to develop as much as possible the deductive thought processes according to the mental model theory. For elementary school pupils this is merely a minor part of natural reasoning. Often they need the use of content-specific rules and causal explanations. A whole new element is formed by final explanations. Pupils tend to add their own beliefs to their reasoning, to include some principle and reach a situation-specific conclusion. This is not valid deductive reasoning, but it may help the pupil to shape and examine his or her own mental model. There is indeed a need in mathematics teaching to constitute a broad common view of accepted premises (norms) on the basis of which we proceed towards theoretical mathematics. Both deductive reasoning according to the mental model theory and also content-specific reasoning that complies with general rules are suitable for mathematics teaching according to constructivism. Mathematics teaching provides a natural context for the development of both forms of thinking without involvement in unnecessary formalism.

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Learning Planning Skills in a Legologo-Based Environment[1]

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The purpose of the present study was to promote the mediation of the thinking skills and heuristic processes of an expert in connection with working in a complex technology environment. The object of learning was an automatic washing machine, the structure and functions of which were simulated by the means of LEGOTCLogo. In order to organize the learning process the central starting points of the theory of cognitive apprenticeship and the constructivist conception of learning were applied. The progress of the washing program of an automatic washing machine was described as a situational graph. The project of 9 hours attempted to make the pupils learn planning skills, and, in order to evaluate the development of these skills, the implementation of the pupils' plans into Logoprograms was analyzed. The criteria for measuring the development of the skills were the hierarchy of the Logo programs and the number of nodes in each program. In addition to this, the progress of four groups of pupils was also analyzed closely on the basis of real time video material, gathered by online -techniques. On the basis of the pupils' programs and the ways in which they were produced it can be concluded that cognitive apprenticeship, connected to the externalization and development of the pupils' own theories and conceptions, is able to support the mediation of planning skills and include them in the pupil's own action effectively. As a result of the project of 9 hours the programs of the pupils were relatively complex and organized; they even modelled the functions of a real-life automatic appliance well.

1. Introduction

During recent years the context of learning has been described as one of the most important problems of learning in school. There have been repeated claims that the relation between the knowledge of the school and that needed in everyday or working life is questionable. On the basis of a number of studies it appears that learning in school does not seem to be able to provide pupils such thinking skills and information that would enable any practical applications of learning. This has resulted in an attempt to place the development of the pupil's thinking and

[1] This article is partly based on a paper presented at the conference of EARLI in Turku, August 1991.

problem-solving skills as one of the most important goals of learning in school (e.g. Resnick 1987, DeCorte 1990).

The critique of the selection of the content of school studies has emphasised the fragmentary nature of the contents of any school subject. The case seems to be that pupils do not realize and understand the structural nature of knowledge, i.e. the interconnectedness of the facts. As a result of this, pupils, when required to analyze new situations and phenomena, face difficulties when trying to memorize and apply what they have learnt. Therefore, all facts learnt are prisoners of their contexts.

The above situation has been further complicated by the technological changes of working and everyday life. The whole world with its phenomena and problems is shown to us as multidimensional and complex. Recently it has been noticed that new skills to outline and organize complex situations are needed, and thus the importance of metacognitive thinking has been stressed. Furthermore, the skills of being able to plan one's own action and being able to identify larger wholes and connections between facts are also mentioned among these new skills (Norman 1980, DeCorte 1990).

Simultaneously with remarkable development in the field of technology, our understanding on the experiences involved in learning has been found to deserve criticism and even be in need of revision. Logo has been one of the new challengers for already two decades (cf. Papert 1980), and more recently there have emerged various new technology environments based on video disc or hypermedia (e.g. The Cognitive and Technology Group at Vanderbilt 1990, Thornbug & Pea 1991). The central aim of these is to produce better simulations of the context and the situations in which the knowledge and skills learnt will be used than previous environments have been able to provide.

However, there remain many questions such as how, in general, to choose the learning environments and ways of learning so that what has been learnt in school can be applied to everyday life? It can also be asked how it is possible to guide the learning process so that its results would consist of organized knowledge and skills of learning which, in turn, would be of use when new, more complex situations and contexts arise in the future. One of the intentions of this study is to explain some of the possibilities of complex technology environments to contribute to the learning of knowledge and planning skills. In this study, a modernized version of traditional apprenticeship, i.e. cognitive apprenticeship, functions as the basic theory of learning guiding the organization of practice.

2. The theoretical framework of the study

It has usually been difficult to outline and make the pupils "see through" the appliances of modern technology. These appliances systematically combine knowledge of many disciplines, which partly explains their complex character. In addition to this, they have become a part of the everyday culture of the pupils in the form of different automatic appliances they daily face at home and in the street. Therefore, it can be thought that if we make the appliances of technology transparent and controllable for pupils, this process might be of help when the different skills connected to thinking and action are linked with the situations and contexts that are important to the pupils.

This study has searched for starting points for learning in a complex setting in the results of modern studies of learning. According to the cognitive conception of learning, the theories or concepts that the pupil possesses about the object or phenomenon under study form the basis of successful learning, and, therefore, they should be taken into account both in the choice of the object of learning and the planning of the learning process (e.g. Resnick 1989). On the other hand, in order to develop into a successful problem solver, a pupil should be able to improve thinking skills of high level and heuristics (e.g. DeCorte 1990). The ability to plan one's own action is often mentioned as one of the most important skills of a successful performer (cf. Glaser 1987). However, the problem of teaching, how to convey thinking skills connected to high-level performance in practical learning situations, has remained unsolved. Consequently, a revised form of traditional apprenticeship, cognitive apprenticeship, which stresses the situational character of learning has been introduced (Collins, Brown & Newman 1989). Next we shall examine more closely the above mentioned views on learning, central in the theoretical background of this study.

2.1. Representations of concepts and mental models in direction of learning

A successful learning process seems to be based on the knowledge the pupil already has and the skills of thinking he/she possesses. Pupils usually have theories of their own, but, if they lack them, in each learning situation they tend to construct theories about the object of learning, may that be a phenomenon or a fact. The representation of concepts the pupil has in his/her possession, his/her mental model, functions as the representation of a theory or an idea. At the same time, this model is a tool that the pupil uses to organize situations which he/she finds similar in some respect. Furthermore, this tool is claimed to help the pupil to identify the

phenomenon or the situation (e.g. Gentner & Stevens 1983; Larkin 1983, 1985; Chi, Feltovich & Glaser 1981).

According to Norman (1980), a mental model of a certain phenomenon or an object has certain typical characteristics: it is not complete nor is it able to fully describe the phenomenon. It might also be difficult to use since some parts of it are easily forgotten. Furthermore, mental models are only seldom based on the results of science or scientific research. People also have a tendency to avoid the use of mental models, if they are able to deal with the situation in other ways: instead of involvement in mental struggle, they might prefer concrete actions.

The approach that stresses the importance of the pupil's own theories, the constructivist point of view, treats the pupil as the subject of the learning process: he/she has an active role to play. This questions the traditional idea of learning, according to which learning is a process in which knowledge is conveyed, communicated, or transferred to the pupil. If we accept the constructivist point of view, even the learning situation should support the externalization of the pupil's own ideas which he/she should also learn to reflect on and criticize so as to be able to develop and even change them.

As the pupil's own ideas and theories function as agents organizing new information, their situations and associations are important, too. If it is possible to connect a new idea or problem to another one, already experienced and lived through, and which can be easily memorized, the processing of the new problem might turn out to be considerably easier (cf. Carey 1986, Resnick 1989, Saariluoma 1990).

The pupil's experiences and feelings are connected to his/her emotions and motives, as well. Therefore, if we allow the pupil's own theories to participate in learning, we might find that these contribute to the structuring of self-piloting learning situations and, in this way, even make the pupils take responsibility for learning. As a consequence of this the every-day life of the pupil, on which he/she reflects his thinking and cincumstances, should be taken into account better and more systemically in both the organization of learning situations and the choice of objects for learning.

2.2. Skilful problem-solving

The learning situation can also be seen as a problem-solving process, in which the pupil attempts to structure his/her understanding of the object or phenomenon to be studied on the basis of his/her premises. In a case of this kind, the pupil's own theories and skills function as the basis of his/her thinking; the goal of thinking, in

turn, is to modify these theories and ideas so that they are as close as possible to those presented by science or the discipline in question. In order to bridge the gap between the point of departure and the goal, the pupil utilizes both thinking and the processes intended to promote problem-solving as much as he/she can.

Above we pointed to the central role allotted to organized knowledge in the process of identifying a successful situation or knowledge. It is possible to compare the functioning of well organized knowledge to a puzzle being solved: it is not possible to place pieces on the table in a random order, but attention must be paid to their interconnectedness, characteristic of puzzles and their pieces. Every time a new piece is put on the table, there arise new possible problem situations and possibilities, which, however, share a context: the picture of a completed puzzle. However, every time a new piece is joined in the puzzle, there are fewer possibilities left. In this respect the situation bears great resemblance to an ordinary problem solving process, in which one has to identify a possible way of going further. As time goes by, or as the puzzle is being completed, we try to choose and insert those pieces that complete the paths already started; the direction of completion is the direction of identified goals. To be a successful puzzle solver one needs, besides both flexible application and solving, conscious choices of different alternatives.

However, a structural identification of situations and phenomena is not enough for a successful problem-solving process: research of these processes has clearly stressed two conditions as the remarkable and necessary conditions of successful problem solving: first, an ability to exploit heuristics and, second, the central role of the control of solution strategies (e.g. DeCorte 1990, Schoenfeld 1985, 1987). Among the heuristics mentioned most often are an ability to plan one's own action, an ability to work forwards or backwards with the problem, the division of problem into subproblems, and the so-called analogical thinking. Examined more closely, to work forwards with the problem might also manifest that the person in question is able to plan his/her own action and identify the path toward the goal. However, it should also be noted that tendencies to take the problem back to earlier, well-known and concrete situations are often defined as the characteristics of the thinking and action of so-called novices (e.g. Chi, Feltovich & Glaser 1981; Chi, Glaser & Farr 1988).

Skilful problem-solving is connected to other factors as well: it is restricted by the limited memory capacity the humans are able to use when processing information. According to contemporary estimations, new and old knowledge is integrated in working memory with a very limited processing capacity (perhaps only 4 chunks). This might easily lead to both mistakes in actions and difficulties

in the construction of a mental model or the representations of concepts. (Saariluoma 1990).

If a person intends to apply knowledge and solve problems successfully, he/she is required to be able to structure a representation of concepts and also to develop it. The difference between skilled and unskilled problem-solvers is revealed in the fact that the skilled problem-solvers are able to accommodate larger representations of concepts and wholes in their memory. On the basis of these, they are able to build larger solution models and plans in order to interpret the situation and, in turn, to solve the problem (Chi, Glaser & Rees 1982, Saariluoma 1990). When required, people are able to develop numerous imaginative and effective ways such as memo lists, diagrams, and drawings to organize knowledge and to facilitate its reproduction.

It should be noted that the necessity to relieve the so-called cognitive burden, referred to above, has usually remained neglected in teaching: it is possible to present the hypothesis that it is useful to teach the pupils to utilize different techniques that help them to remember and memorize better (cf. Kotovsky, Hayes & Simon 1985). These are not the only reasons for the teaching of such techniques: it should be remembered that graphic facilities bear remarkable similarities to the techniques used by professional planners. Therefore, there seem to exist many good reasons for teaching to include the use of different facilities intended to organize knowledge and thinking: examples of these are graphs, tree diagrams, and semantic networks, all of them promoting operations for both teaching and learning. It should also be noticed that they bring the working and everyday life closer to school work (cf. Rubinstein 1986).

2.3. Planning as a strategic skill

It is difficult to give a thorough definition of the concept "planning" since it is used in connection with a number of different and even controversial cultural and social contexts (e.g. economics, administration, politics). Usually planning is identified with problem solving, or it is at least seen as its necessary part.

In the context of learning, planning has been viewed from two standpoints: it is seen either as a general cognitive skill or as an operation connected to a certain and specific content (Scholnich & Friedman 1987). Some theories characterize planning as one of the most essential and important factors in problem solving, whereas others wish to stress the difference between them. The definitions of planning almost always include a target, the operations connected to it and the

operations intended; the definitions usually claim that planning is supported by the same concepts as problem solving.

According to Kreitler and Kreitler (1987), there is a difference between planning and problem solving: the former is a form of cognitive construction based on one or more steps, whereas the latter consists of the cognitive performance of the results of the prior construction process, the plans, and the evaluation of their practical applicability, perhaps linked with concrete manipulation. Furthermore, planning is always connected to action that will take place in the future, whereas problem solving is not necessarily linked to action or future. In addition to these differences, the focus of planning is most often on the fact how to achieve certain goals.

Plans are results of planning processes which can be divided into three major phases (DeLisi 1987):

1) The recognizition of a need for a plan.
2) The making of a plan.
3) The fulfilment of a plan.

Plans and planning are often described as a group of complex conceptual skills, connected with a number of factors: reflection of knowledge, abilities to present knowledge in a flexible and general manner, an ability to recognize and set goals, strategic skills, and abilities to monitor, evaluate and correct the strategies in use. Therefore, it can be claimed that a skilful planner has to know that he/she knows how to plan (DeLisi 1987).

A plan always includes a description of the goals and procedures needed to achieve the goal in question. The form of a plan is most often a figure, a table or a program that shows the objects, times and places of the intended actions. One possible criterion to notice and evaluate possible development in planning skills could be to study skills needed to present an external plan. In teaching, planning skills could be developed by paying more attention to the teaching of strategies and ways of presenting organized knowledge.

2.4. Differences between novices and experts

Studies of the differences between the knowledge and behaviour of novices and experts of various fields of life (e.g. chess, physics, radiology) in problem situations have revealed many new observations supporting each other that also touch upon learning (de Groot 1965, Larkin, McDermott, Simon & Simon 1980, Schwartz & Griffin 1986, Glaser 1987).

Our knowledge structure seems to depend on the ways in which we represent the problem in question. Novices, on the one hand, tend to organize their representations on so-called surface information and that information they can directly take from the problem. Experts, on the other hand, base their knowledge on those principles and abstractions they are able to construct on the basis of the problem. There are other differences, too: the knowledge the expert possesses is very procedural and goal oriented. This kind of knowledge is easy to utilize and possible to control, mainly because of the automatization of the operations, achieved as a result of experiences.

Studies on expertise have mainly focused on such content areas where the problems to be solved tend to be well defined, which has also been the most criticized point in the results these studies have produced. The critics maintain that most of the complex situations and problems requiring expertise are actually the opposite of well-defined situations, they are ill defined. Theories on the decision making processes of an expert seem to deviate from each other; they also seem to be closely connected to the context of the problems an expert faces and the content area of his understanding (cf. Dreyfys & Dreyfus 1986, Glaser 1987). A further target for critical remarks has been the methodology involved in expertise studies (Willson 1990).

The problems and contents of expertise studies are often closely connected to the contents of the school. Hence it is possible to argue that the results of expertise studies might be of significance to the development of learning in school. Even though teaching for its part has, even until now, attempted to develop the behaviour of the pupils into an expert-like direction, expertise studies do have new opportunities, especially in the reformulation of cognitive goals for learning. Their results, however, will not be able to act as bases on which direct conclusions for the organization of teaching can be drawn.

2.5. Cognitive apprenticeship

More and more tasks in working life require the worker to apply high level thinking skills - sometimes connected with physical efforts. Professional tasks challenge us to utilize structural knowledge connected to the context of tasks and to apply thinking and problem solving skills involved in our tasks. This process is in progress especially in such fields where information technology has become a part of the learning process.

Most often the reorganization of activities has been based on the main principle according to which learning and thinking should be united with the

operations they are connected to. There are a number of reasons for this: learning is both contextually bound and of constructive character. Furthermore, even the practical problems we face do require that the above described thinking should become more generally taken into account. It can be argued that the solution of the complex problems of both everyday and working life will be extremely difficult, unless these problems are chosen as objects of study already at school and the pupils do not learn to solve them or at least parts of them. In practice, this is possible through the organization of the curriculum so that thinking, learning, and action are united in time and place. This, in its turn, requires that learning situations are reorganized into long-term, self-piloting projects. However, there remains the problem how to organize these situations so that an integral part of our culture, strategic behaviour of an expert, can be successfully conveyed to the pupils in connection to these projects.

The basic idea of cognitive apprenticeship theory is to link a traditional and successful way of learning different skills and the results of modern cognitive learning theory (Collins, Brown & Newman 1989). This theory has been planned in order to teach the processes the experts use by allowing the learner see the socalled silent knowledge the experts utilize. In order to achieve this goal, both thinking and knowledge are exemplified and shown in their context: learning is mostly based on experiences, and the role of cognitive and metacognitive processes is stressed. The theory of cognitive apprenticeship attempts to merge learning, thinking, and doing into extremely life-like contextual connections that simulate the use of knowledge structure.

Cognitive apprenticeship applies four different teaching methods: modelling, coaching, scaffolding and fading, and, finally, reflection.

In the case of modelling the pupil starts by watching the teacher solve the problem. The teacher progresses slowly and stresses the central elements involved in the process. If the pupil were to learn by his/her observations, he/she must be able to infer, to co-construct, the goal that the teacher is striving for. Therefore the teacher should think aloud during the process: this idea is based on the hypothesis that, when doing this, the teacher promotes the structuration and sharpening of the pupil's mental models and representations of concepts; this also develops the use of metacognitive thinking during the learning process.

During coaching the teacher helps the pupils when needed so that they are able to fulfil all their assignments. If needed, the teacher might advice them directly or he/she can also support the pupil's independent and active work through the tasks.

Scaffolding and fading is, perhaps, the most demanding and central of the approaches in the use of the teacher, since the teacher is required to be able to

utilize the skill of applying strategic thinking. In practice the case might be that the teacher tries to get the pupil started on the task by giving him/her small hints and other forms of indirect help, and, as the operations of the pupil become more secure, the teacher should fade away. However, if the pupil faces extreme difficulties, the teacher returns to the role of the coach.

Reflection is the second of the indirect approaches of the theory: the pupil compares his/her performance to that of the teacher or another pupil in order to develop his/her own ways of operation. In this way the pupil must reflect his/her knowledge and thinking on activities which have more expert-like features. It often happens that the roles of the teacher and the pupil are inverted at this stage: the pupil becomes a reflector of his/her own actions and thinking, he/she reorganizes them and reveals them to the teacher who, in his/her turn, listens to the pupil, asks strategic questions, or gives hints.

Cognitive apprenticeship has been applied successfully to learning in a number of different fields: writing (Scardamalia & Bereiter 1985), reading (Palincsar & Brown 1984), and problem solving in mathematics (Schoenfeld 1985) are good examples. Furthermore, it has been applied to the learning of the use of complex technical machines (Lajoie & Lesgold 1989) and the strategies of decision-making and reasoning (Thornburg & Pea 1991).

3. Research questions

One of the aims of the study was to apply cognitive apprenticeship to the learning of thinking and knowledge involved in complex phenomena. At the same time a series of learning operations were created that would not only take the conceptions and mental models of the pupils into account, but regard them as the premises of learning. Furthermore, the facilities connected to the learning process were meant to support the process intended to develop expert-like behaviour.

The structure and ways of operation of a real-life automatic device, the automatic washing machine, was modelled by using a graphic way of organizing knowledge, the tree structure. The central didactic methods of cognitive apprenticeship were applied to the actual working in a complex technology environment; LEGOTCLogo served as the modelling environment for the structure and functions of an automatic appliance.

The central research question was, (a) to explain what kinds of plans the pupils would be able to provide for the operations of an automatic washing machine when participating in a project work of about 6 hours. Furthermore, (b)

we also wished to find out what kind of strategies the pupils utilized when implementing the plans.

4. Method

4.1. The learning environment

A number of different reasons led to the choice of the automatic washing machine as the learning environment: first of all, as the washing machine is an automatic appliance familiar to each pupil, they already possess a mental model of it. As there probably is a washing machine in their homes, the construction of the mental model might be helped by experiments and further studies at home. Furthermore, it is possible to structure the practice of teaching so that it simulates the situations and activities in real life to a great extent. This, in its turn, makes it possible for the pupils to examine the knowledge they have acquired in its original context.

Besides these reasons, the different types of knowledge, both declarative and procedural and knowledge related to the control of operations, involved in the structure and operation of an automatic washing machine are highly organized, and, therefore, possible for novices to model. The automatic washing machine is a typical example of control technology, a part of the technical culture of today's everyday and working life. By introducing its structure and operations to the pupils it can be assumed that they also become familiar with an essential part of the knowledge of today's culture effectively.

LEGOTCLogo was used to model the automatic washing machine: in this environment the model is first constructed from lego blocks and then programmed to operate by the Logo-programming language. One of the reasons to choose lego bricks as a part of the project was that they are highly concrete; in addition to this, all of 13-14-year-olds involved in the study were familiar with them. It was also assumed that the pupils had experiences and mental images which might contribute to their thinking, and be of great help in their planning and problem solving.

4.2 The organization of the project

The project of control technology consisted of three successive classes, lasting three hours each, during three weeks. The last class was the most important one in the gathering of data for the research. In the beginning of the project the pupils worked

at home in order to examine the structure and operations of the automatic washing machine.

During the first proper class there was a discussion led by the teacher, on the mental models the pupils had constructed on the basis of the work at home. In the same situation the structure of the washing machine and the phases of the washing program were being modelled by the help of a tree structure and a flow chart. Soon after this the pupils started to build a model of the automatic washing machine by using lego bricks. This phase was supported by an instruction booklet.

After having completed their models the pupils connected them to their computers and started to examine the operations of the model and, at the same time, to study the Logo programming language. In order to make this possible, the pupils had received written material to work on. Roughly speaking, the aim of this was to learn how to make the model function by giving commands to the different parts of the model such as motors, lights, or sensors.

Having learnt the basic commands of Logo and after starting to see the connection between the commands and the model, the pupils started to plan how to program a whole washing program. This took place so that the teacher, by discussing with the pupils and by thinking aloud, modelled the structure and actions of the washing program once again, but this time he used a situational graph (cf. Boy 1991, Rubinstein 1986). The ways of describing the washing program through Logo-language were outlined. Partly because of the need to relieve cognitive burden, it is only natural and justifiable to describe the structure of a program with the help of graphs before the actual implementation is started. Furthermore, in this study the use of graphic techniques functioned as a method intended to develop planning skills. The connection between the tree structure and the Logo-program guiding the operations of the model can be seen in Figure 1 on the next page.

After planning and its implementation into the Logo-environment had been modelled, the pupils started to structure their own programs, working in groups of two. During this phase the teacher attempted to support learning in different indirect ways in order to enable further progress. Similarly, as the pupils learnt to program, the teacher gradually faded away. When problems emerged and the teacher discussed with the pupils, he attempted to stress the development of metacognitive thinking and the reflection of thinking and action by questions such as the following ones:

Tell me what you are doing.
What is your goal?
How should the model behave?
What have you done up to now?

What kind of parts does your program include?
How should your program be changed so that it would do this and that?
How does your program work?
Explain it to me right from the beginning, all the way to the end.
What stage are you now at?
How does the program work if you change the place of these procedures?
How is the program connected to the tree structure?
How would you explain to your friend the way in which your program works.

Main procedure
to coloured washing
prewashing
washing
rinsing
spinning
end

Subprocedure 1
to prewashing
water in
rotation
water out
end

Subprocedure 2
to rotation
clockwise
change of direction
anti-clockwise
end

Subprocedure 3
to clockwise
talkto "a onfor 30
etc.

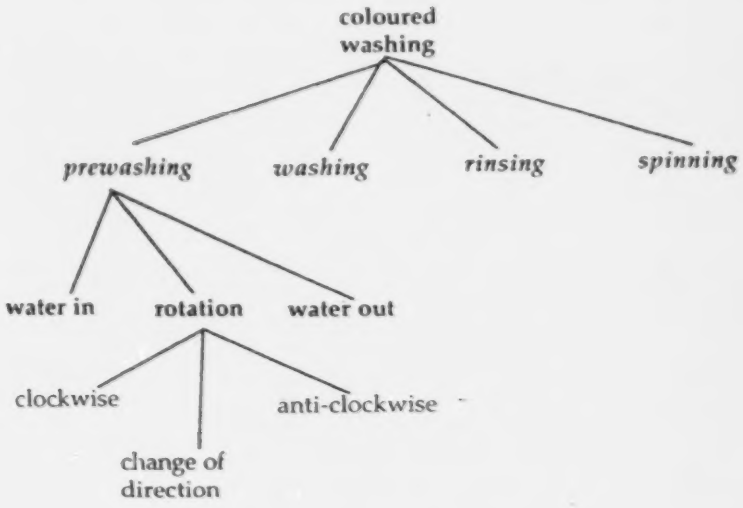

FIGURE 1. *The modelling of the washing program for coloured washing of an automatic washing machine by means of a tree structure and programmed by TClogo.*

In the beginning of the last class the pupils were given the actual project assignment, which functioned as a test in this research project. During the fulfilment of this task the teacher tried to stay in the background, away from the pupils and their tasks. The pupils had almost 3 hours to complete the project task, and its goals can be seen in the description below:

Plan and program a model for an automatic washing machine. It has to fulfil the following conditions:
The model must correspond to your conception of the procedures of some of the washing programs as well as possible. It should include at least four phases, all of which must differ from each other. The program is not allowed to start until the door is closed. It should also give a sound signal when it leaves one phase and starts the next. When planning, you can use the help of a tree structure. When the program works as you had thought it should, you have completed the task. Finally, save the program and document it in your notebook.

4.3. The subjects of the study

Totally 26 pupils participated in the project. There were two main groups, one consisting of boys, one of girls. The reasons for choosing this kind of selection stemmed from practical difficulties in organizing the test in any other way. However, homogeneous groups made it possible to compare the girls' and the boys' ways of accomplishing the tasks, though this was not an original aim of the study. From the point of view of the pupils' success at school, both groups were carefully selected to include pupils of all levels. Therefore the groups can be seen to represent "normal" classes.

All pupils in the project had worked on a computer before; 90 % of them claimed to have worked on lego bricks before. 4 pupils out of 26 had even earlier been introduced to the basics of Logo (turtle graphics). In practice Logo was new to all pupils.

4.4. The gathering of data and the principles of data analysis

Answers to the problems of the study were sought by analyzing the program listings that the pupils had saved on computer discs during their tasks. Thus it is possible to assess what kind of plans the pupils had constructed during the accomplishing their tasks. The analysis of the program listings was done for 13 pairs of pupils.

The examination of the strategies involved in the fulfilment of the plans is based on the analysis of videotaped material. This made it possible to clarify the talk of the working pairs and of the teacher, the contents and the nature of the possible discussion between them during the progressing phase, and also the flow of displays in real time. Therefore, by using so-called on-line techniques it is

possible to obtain exact knowledge, related to the learning process on how the pupil's actions proceed. The videotaped material allowed a close qualitative analysis of the accomplishment of the task during the phase when problems were being solved. However, it was possible to clarify the strategies of two pairs of boys and two pairs of girls only, since the number of video cards in use was limited.

The analysis of the program listings of the pairs allows a post-task construction of the ways in which the pairs arrived at their plans as the outcome of a short theoretical phase and programming. Since the role of the teacher during the project phase was to stay away from the projects of the pupils, the final programs can be seen as the pupils' own programs, to represent their thinking. It should be noted that the teacher tried to give as short answers as possible; in cases of deadlock the teacher referred to a possible source of information.

The results of the analysis of the outlined plan will be described by using a similar tree structure that the pupils had been systematically trained to use during the phase of programming.

The analysis of the video material is based on the following phases: after an overall viewing everything that was to be seen on the screen during the pupils' accomplishment of the task was written down. Furthermore, all tests and corrections that occurred during the construction of the program code were also recorded. By examining the actions and testings that occurred during the phase of implementing the plans, it was relatively easy to identify the strategies the pupils used to fulfil their plans.

5. Results

5.1. The realizations of the plans

The model of the washing program the pupils were being mediated was a wellorganized system, consisting clearly of functions in different phases and their subfunctions; it was also supported by a situational graph. In an ideal case a situational graph should consist of a root and at least two further levels.

The analysis of the washing program in Logo-language was based on a tree structure constructed on the basis of the programs of in all 13 working pairs. The table below concentrates on the final programs: it shows the numbers of the nodes programmed (subprocedures) and the number of nodes on hierarchical levels. Girls and boys are shown separately.

TABLE 1. *Programs produced by pairs of pupils and the number of their hierarchical levels and nodes.*

	Pair	Number of nodes	Number of levels
Boys			
	A	21	3
	B	14	1
	C	10	1
	D	12	2
	E	9	1
	F	5	1
Girls			
	G	9	1
	H	10	1
	I	7	1
	J	9	1
	K	5	1
	L	6	1
	M	21	2

We can see that only less than one quarter of working pairs (3 out of 13) produced programs with more than one level. All pairs were able to identify that the washing program consisted of a linear chain in which a number of phases followed each other, but only a few pairs realized that these phases are based on subphases, and, furthermore, that these subphases might be similar and used by all the main phases of the washing program (prewashing, washing, rinsing, spinning etc.). However, only one pair (M) was able to utilize thinking developed to this stage. The examples (tree structures) below describe the results of the plans of two pairs: the plan of pair A includes three well organized levels, whereas the plan of pair G is typical of those plans that consist of only one level.

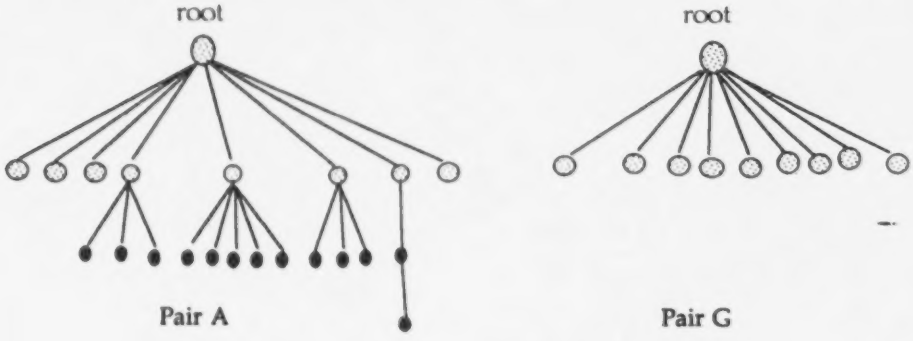

FIGURE 2. *The fulfilment of the plans of two pairs depicted as hierarchical tree structures (● = phase and ● = its subphase).*

Usually the procedures of the pairs give an organized picture of the plan intended to function as a washing program. The programs included some repetition of certain functions such as circles back and forth and most of them had been programmed by using the "repeat" command.

The programs of the boys include somewhat more nodes than those of the girls: the average number was 10 in the programs of the girls and 12 in those of the boys. All washing programs included at least four phases. The plans of two pairs of girls described closely the progress of the actual washing phase; two pairs of boys were unable to make their program to check the position of the door.

To sum it up, the case seems to be that even a short project work such as this is able to provide the pupils with a good understanding of the structure of the washing program and the procedures of the program. The pupils were also able to describe the different phases of the program in a correct order, modelling a real washing program very well.

5.2. The strategies of programming

It is possible to consider the process of fulfilling a plan for a washing program as a representation of a process of solving a problem or as a search for information which can be structured by a situational graph on which the planning of the washing program was based. This study utilized a tree structure in order to give a graphic account of the representations of the situations.

In the study of artificial intelligence (AI) the search for information taking place by the help of situational graphs is called chaining. If the procedures intended to fulfil the plan start from the root of the tree, the main procedure, and progress to its leaves, we can talk of forward chaining. Similarly, if the problem solving procedures start from the original problem or the goal, proceed through its subproblems or intermediate goals so that the subproblems can be solved by the help of primitives, this form of chaining can be called backward chaining (Boy 1991). Forward and backward chaining are closely related to the so-called bottomup and top-down methods applied to program a computer to solve problems.

The graphic model provided for the pupils stressed the need to divide the final goal into intermediate goals and, therefore, to give a well-organized representation of the washing program. Consequently, it was assumed that the pupils' programs would have shown a remarkable emphasis on backward chaining. Next we shall focus on four pairs and their progress in tree structures.

Pair L.

Figure 3 below shows the result of this pair, consisting of two girls, in the form of a tree structure. The same figure will also show how the fulfilment of the plan proceeded, and, furthermore, which parts were tested in order to check the direction of progressing.

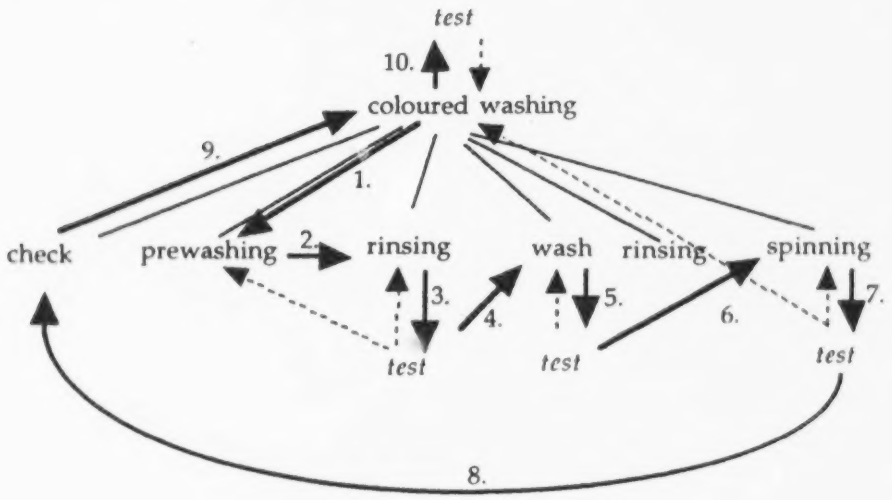

FIGURE 3. *The progress of the programming process of pair L (numbers 1., 2., 3., etc. describe operations in consecutive order).*

The program started by defining the main procedure, which was followed by a description of its subprocedures. These were being tested, too: two first ones when they had been finished, and, after this, the new ones by themselves or together with the main procedure. The procedure of checking the position of the door was defined and added to the main procedure in the end of the task. The first test for the main procedure did not occur until the last subprocedure (spinning) had been completed; the second test occurred after the procedure of checking the position of the door had been defined and joined in as one of its subprocedures.

This pair worked systematically without any special problems. On the basis of the organization of their process-like work and the videotaped discussion it can be assumed that all the time they had in their minds the goal toward which they worked. This idea is supported by the fact that the tests of the sub- and main procedures took place in parallel to each other. They also concentrated their attention on the task and talked only about the actions they were taking.

In the beginning the fulfilment of the plan was backward chaining since the main procedure was defined by its subprocedures. However, the pair was either not able or willing to divide these into further parts, but the subphases of each phase were described on the level of Logo-primitives. On this phase the fulfilment of the plan took place by following forward chaining and thinking. The final result was a well-organized model, though only of limited depth, and its description intended to function as a washing program.

Pair M.

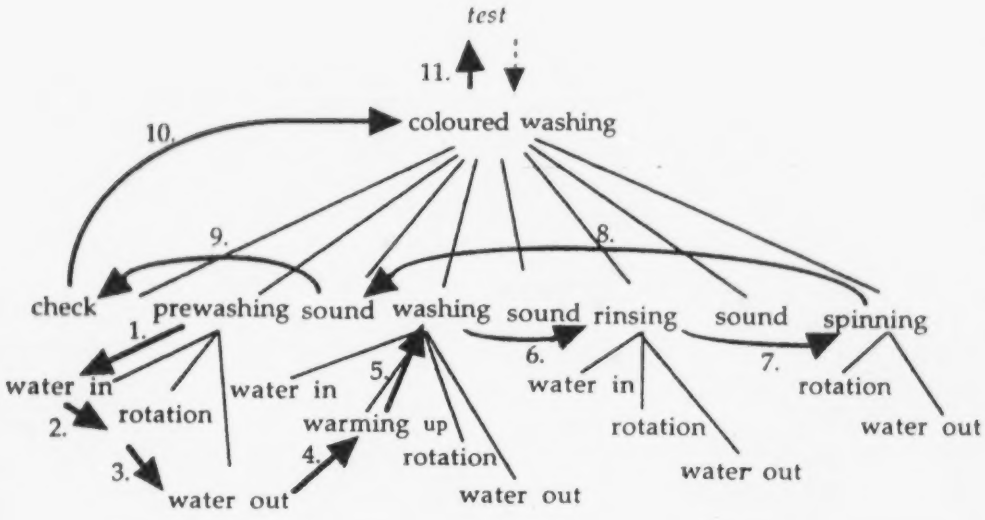

FIGURE 4. *The progress of the programming process of pair M (numbers 1., 2., 3., etc. describe operations in consecutive order).*

The work of this pair of girls resulted in a well-organized program corresponding perfectly to the model of the structure of the washing program that the pupils had been provided during the modelling phase. Furthermore, even their plan for the washing program was well articulated before they started the actual programming, which is revealed through the certainty they showed in the fulfilment of their plan: all experiments and tests were restricted to the end of the programming session.

The activity of these pupils can be described as a fully developed backward chaining: they were able to divide the goal into subgoals and even their subgoals until these could be carried out as Logo-procedures consisting of one line.

The fulfilment of the plan proceeded so that they first defined the procedures as deeply as possible, and, at the same time, paid attention to the fact that the subprocedure can be used to describe other phases as well. The definition started by describing the prewash and proceeded systematically to the writing of the main procedure.

The procedure occurred exactly like the search for information in the depthfirst method of artificial intelligence (AI). This approach guaranteed this working pair an economic and almost errorless path to carry out their plan.

Pair F.

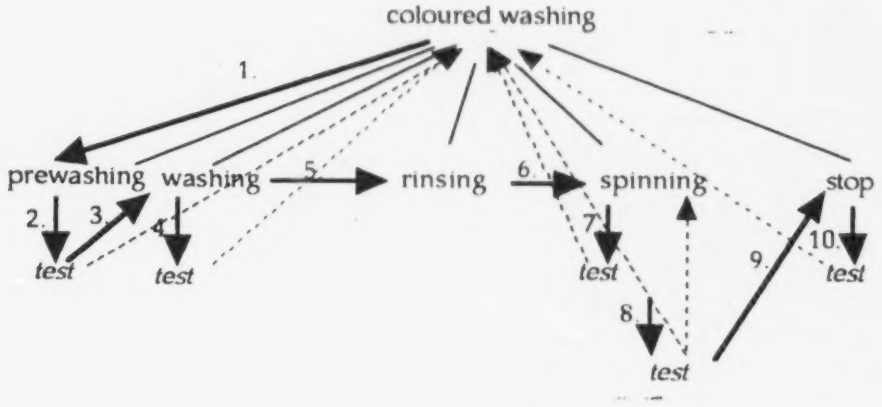

FIGURE 5. *The progress of the programming process of pair F.*

As we can easily see, the procedures of this pair differed from those we dealt with above: the procedure of pair F were based on experimentations. As soon as they had defined the main procedure, they continued by defining the subprocedures. After each subprocedure was completed, it was tested. Contrary to our expectations, the pupils decided to test the functioning of the main procedure before testing the correctness of a possible subprocedure.

As the pupils proceeded further, they faced difficulties: this was easily seen in the definition of the spinning procedure which required a number of experiments and tests. These difficulties and the constant habit to test the function of the main procedure indicate that the plan of pair F was unarticulated before they started to program. As a result of this, they were at the mercy of Logo and in need of tutoring (I don't know how to ...) when trying to carry out the plan.

However, their final result was quite successful and describes the phased proceeding of the washing program satisfactorily, though it does not exactly resemble those conceptions that the teacher had tried to convey to the pupils during modelling. When fulfilling the plan they proceeded from main procedures to subprocedures, and the definition of the latter was carried out by the use of Logo primitives. Their way of proceeding can be characterized as one reaching limited depth.

Pair E.

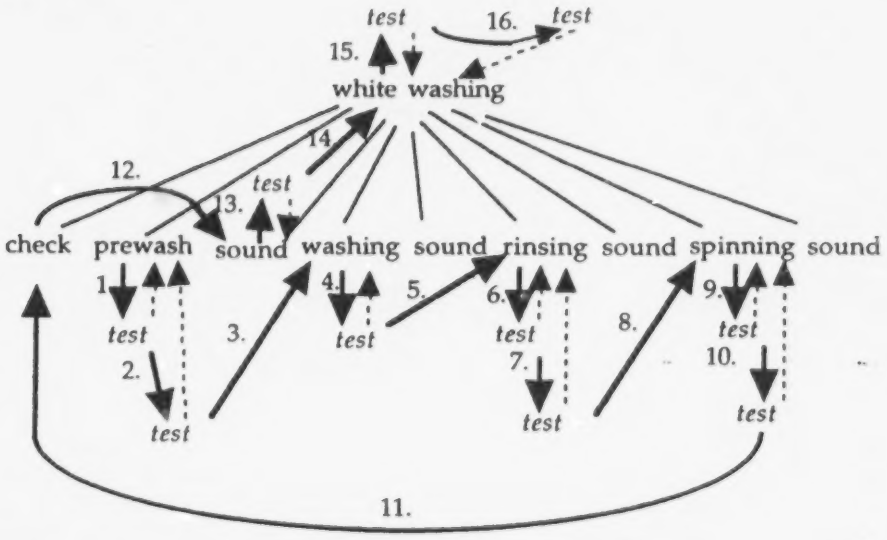

FIGURE 6. *The progress of the programming process of pair E.*

As the figure shows, this pair worked systematically after having cleared some initial problems. The programming was started by prewash that was followed by one subprocedure after another, first programming and then testing.

These pupils had several problems to make their program function. In figure 6 you can see this in the cases of prewash, rinsing, spinning, and white wash: each procedure required usually two tests and corrections to work properly. However, their mistakes were mostly mistakes in writing only. The procedures they produced were quite complex and bore a great resemblance to a real washing program. This fact reveals that these pupils had learnt to express their plans and to describe the functions of a washing program successfully by using Logo.

In the fulfilment of the plan this pair acted like pair L: the goals of the plan had been set to reach limited depth only. When they had outlined the main procedures of the washing program, they tried to describe the procedures within the main procedures by using Logo-primitives, though, from a logical point of view, it would have been easier to proceed by dividing the main procedure into subprocedures and continuing to program them in the manner of pair M.

Pair E started to work by backward chaining, but after the initial phase they acted just as other pairs did. On the level of subprocedures they decided to turn to the opposite direction.

6. Summary

This study has analyzed plans intended to function as the program of an automatic washing machine. These plans have been created and carried out by pupils, age 13-14, in a LEGOLogo environment. The estimations on the level of the plans are based on the ways in which the pupils have described them in the Logo programming language. Besides this, the data collected by online techniques allowed an analysis of the processes of four groups of pupils carrying out their plans.

The conveying of organized knowledge by the teacher in the form of a situational graph seemed to function as an effective means of developing the skills of planning and programming. Only after a few hours of programming, the Logo-procedures functioning as the expression of the pupils' planning were organized and consisted of several nodes (subprocedures) that were interdependent on each other. These Logo-procedures also succeeded in modeling the procedures of a real washing program very well. The case seems to be that an approach based the theory of cognitive apprenticeship is able to convey expertise that many complex objects of learning contain.

As an interpretive language Logo does not naturally support the division of the goal into subgoals nor the problem solving method based on planning that an expert exploits. The present approach in which planning is linked to a situational graph helps to provide for the experimentation phase an organized procedural path that has some similarities to the behaviour of an expert. A similar method has earlier been applied to Logo programming by Verschaffel et al. (1990) and DeCorte et al. (1991).

Only a few pairs of pupils were able to internalize a fully modelled backward way of action as a result of a short experiment. Such pairs of pupils who planned a program with many hierarchical levels of the graph being modelled can be seen to represent the behaviour described. Most often pupils reached the first hierarchical level and, after that, decided to work forwards. However, on the basis of the results it can be concluded that, at least to a certain extent, it is possible for the pupils to learn working directed backwards as a result of a short-term practice, provided that they are guided into that direction systemically, by utilizing the methods and theory of cognitive apprenticeship.

It is our opinion that pupils should already during their school years be prepared to examine complex phenomena and to solve insufficiently defined problems connected to these phenomena (cf. Achtenhagen 1990). One of the aims of this study was to develop ways of action that can be used in learning in school in order to approach complex phenomena that touch upon the fields of different academic disciplines. By considering and analyzing them it is possible to connect learning to thinking and operations, which also results in a very situational object for learning. On the other hand, values, attitudes and even ethical points of view are naturally united with learning situations.

Control technology forms a complex whole consisting of many disciplines, known to us in everyday life. Through the LegoLogo system the inbuilt knowledge of control technology can be made transparent and, at the same time, the object of the pupils' thinking and studies, developing these in an effective way.

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Curriculum Development in the Finnish Comprehensive School Mathematics - Progress and Challenges

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In the early 1990s a curriculum development work concerning general education (comprehensive school/upper secondary school) was started in Finland. According to the plans the new curricula should be adopted in schools in the middle of the decade. This article describes the curriculum work in the comprehensive school mathematics. Firstly, I will examine the progress of the work on the development of mathematics education, concentrating on a more detailed presentation of the development principles that were created during the work process. Secondly, I will analyze and discuss the problems and possibilities of the implementation of the curriculum plan.

1. Approaching a reform

A strong change process was initiated in many Western countries, when, at the beginning of the 1980s, USA's teacher organization introduced its agenda for the reform of mathematics teaching. Several extensive assessment projects probing the state of school mathematics were also carried out in the 1980s (e.g. SIMS; Travers & Westbury 1989, Robitaille & Garden 1989), whose results created much discussion, as well as some anxiety. Many analyses came to the conclusion that students are not acquiring the kind of mathematical skills and understanding that they will need when they solve the cultural, economic, political, scientific and environmental problems of the future.

In the United States the results of SIMS (The Second International Mathematics Study) produced strong reactions in 1983, and accordingly earnest measures followed. A great amount of resources were invested in the innovation of mathematics teaching, and the US mathematics teacher organization prepared the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) and the *Professional Standards for Teaching Mathematics* (NCTM 1991). In England the motivating force of the reform was The *Cockroft Report* from 1982. During this reform process a new student evaluation system was created and a new national curriculum plan was made (DES 1989). In Japan new curriculum plans were

introduced in 1989 from the kindergarten level to the upper secondary school. Curriculum reforms were also carried out, or are being carried out, in most Nordic countries.

In Finland the changes in mathematics teaching and curricula have usually been repercussions of the reform waves that have emerged in the Western countries (mainly in the USA) (cf. Kupari 1989). In autumn 1987 a very special and significant committee project was started in this country, as regards mathematics and science education (the so-called *Leikola Committee*). In its interim report (Committee Report 1989) the committee presented a thorough evaluation of the state of mathematics and science education giving, among other things, the following statement:

> *"Mathematical-scientific basic skills are generally rather good...Both at the elementary and lower secondary level the teaching is, however, too much concentrated on the enhancement of mechanical computation skills, at the expense of problem-solving and application skills, and therefore mathematical thinking cannot develop appropriately."*

In its final report, which was published about a year later (Committee Report 1989) the committee presented the goals of mathematics and science basic education, brought up important problem areas and proposed measures for correcting shortcomings and defects.

As an almost immediate consequence of the proposals of the Leikola Committee, the National Board of Education set up in February 1990 a *Working Group for the Development of Mathematics Education*. The task of this Development Group was to "evaluate the development needs of mathematics teaching in the comprehensive school and upper secondary school, and to make development proposals", and 12 members were appointed to it from different levels of teacher education, research and teaching.

2. Emphasis areas in the work of the Development Group

The working group approached its task in the spirit of the ICMI's report, School Mathematics in the 1990s (ICMI Studies 1986):

> *"It is to be hoped that those responsible for the 1990s mathematics curriculum in national educational systems will seek to get to grips to such (more extensive) issues, and not to be content merely to tinker with the more superficial details of*

whether this topic or that should be included in the syllabus for a particular school year."

Focal areas in the work of the group can be described, for example, by using the blueprint proposed by Leithwood (1981). Leithwood has described the curriculum as an integrated whole, which contains nine sub-factors: platforms or images, objectives, student entry behavior, assessment tools and procedures, instructional material, learner experiences, teaching strategies, content and time. The working group used a great deal of time for discussing the kind of thinking that should form the basis of mathematics teaching in general education. This involved a thorough analysis of the changes that have taken place in learning and teaching views of mathematics, and the consideration of these changes in the curriculum construction work, as well as in the planning and realization of teaching. There was a great consensus of opinion that it is not possible to bring about the desirable change merely by an adjustment of the content and frame factors of teaching. In addition to these, we must have a vision on what kind of cognitive and learning conceptions mathematics teaching can be based, and show what kinds of learning experiences can be produced on this basis and how.

The evaluation of teaching and learning were seen as a very important field in the development and renewal of teaching. The group considered it important that evaluation should be understood more comprehensively than before, that versatile assessment tools should be developed, and that projects should be initiated for testing the new evaluation practices. Especially in regard to learning materials, new types of solutions were hoped for, in order to initiate innovative material production.

3. Development principles of mathematics education

In May 1991 the working group completed its report, having worked for over eighteen months (National Board of Education 1991). The most central contribution of the report is undoubtedly found in its fourth chapter, which describes the development principles of mathematics education. In the following pages the principles are described by summarizing and grouping them under a few main headings. The presentation is based on articles, which have examined the birth and content of these principles in more detail (Haapasalo 1990, Haapasalo & Kupari 1991, 1992).

3.1 Learning of mathematics and its significance

In constructivism mathematics is understood as the development of thinking processes and as a search for and construction of connections, rules and patterns, starting from the students' actual experiences. During this kind of an active mental process the student constructs for himself a picture of reality. New pieces of information become assimilated into his cognitive system, while at the same time modifying it, in constant interaction with the environment. It is the task of the teacher to design learning environments where the student can develop versatile thinking and action patterns on the basis of his own knowledge, mental images and conceptions.

In the new curriculum plan the starting points and goals of mathematics teaching must be clearly presented. Overemphasis on theoretical knowledge should be avoided and the more general nature of mathematics, which is not dependent on contents, should be taken into account. The learning of paradigm-construction, organization and classification are most important, and they can first take place in strong dependence on contents and situations, without symbolic presentation. From there the teaching can proceed, according to the student's abilities, towards a more abstract and symbolic presentation, which is not dependent on contents.

The significance of mathematics should be seen in a broad sense, as an important influence in the student's mental growth process, which prepares the student for goal-oriented activity and social interaction. The importance of problemsolving processes is seen mainly, not only in the acquisition of knowledge, but also in its application. Making the appropriate questions, an overall perception of the problem, definition of the task, discovery and execution of the right solution processes and evaluation and formulation of the results are central factors in the learning situations. The first three development principles emphasize the abovementioned points, and they actually contain the basis of the cognitive learning conception:

Principle 1. *Mathematics should be understood more broadly than only as the learning of certain computation skills. Acquisition of mathematics should be combined with the student's mental growth, and with extending the scope and logic of his thinking, as well as with the formation of his world view. Problem-solving should be a central teaching principle, in addition to the mathematical and logical requirements.*

Principle 2. *The primary goal of mathematics teaching should be the development of the student's ability to classify, organize, and model situations that confront him in the surrounding world. Consequently, the starting point of teaching should not be*

prestructured information, which is presented as given to the students. Instead, by emphasizing heuristic information acquisition processes, the aim should be that the student's knowledge and skills are gradually organized into useful structures.

Principle 3. *The student is seen as an active acquirer, processor and retriever of knowledge, for whom learning means modification and expansion of earlier thinking and operation patterns.*

3.2 Presentation of the goals and planning of the learning environments

The definition of goals is one of the most important questions of curriculum construction and, at the same time, one of its greatest problems. The importance of the goals is underscored by the fact that teaching and construction of teaching materials are mainly directed by the national *Foundations of the Curriculum*. The general goal statements that form the basis of the present curricula are quite adequate in themselves, but they are largely neglected in practical teaching work. One reason for this is certainly the fact that the general goals and the contents are presented separately, and their connections are not described at all. The mathematics syllabus thus mainly consists of a collection of teaching contents, which have been distributed to the different school years, according to what is thought possible to teach to the students in each particular age period. It is very difficult to combine the process goals with this kind of a list of objectives based on teaching contents: the objectives tend to become contents or the other way around.

In the new curricula the goals must be described so that they give proposals for a variety of approaches and working methods. A more detailed description should be given of how to design the learning environment and its materials so that it supports active, experimental, inquiring and discovering forms of learning.

Principle 4. *Instead of the content-centered goals set for teaching, process goals of learning are emphasized. These should be formulated in a spirit that is supportive of the student's mental growth, so that they also direct practical teaching work and the construction and use of teaching materials.*

School mathematics contains information which, for the most part, is very old; it does not include hardly anything discovered in the present century. This pre-structured knowledge has been taught to the students in schools for decades, mainly by deductive and teacher-centered teaching methods. Most students have had no experience of the fact that mathematical knowledge has been created as a

result of problem-solving processes. Therefore, more important than to preserve the traditional contents is to take into account the perspectives offered to information acquisition and learning by the history of how mathematical knowledge was created. Those mathematical ideas and great discoveries that have promoted science and its applications should be brought to light, where they have been dominated by excessive computational practice.

Principle 5. *Mathematics must also be open in its contents for new information, discoveries, significant new contexts and current applications. Although the birth history of mathematics should be taken into account in structuring the teaching contents, we should have the courage to examine critically the traditional learning contents. We must be able to exclude such information that is not absolutely necessary for understanding the structure of mathematics.*

Mathematical applications and problem situations should be sought also in other subjects than in the natural sciences. Especially on the lower stage of the comprehensive school connections with other school subjects should be utilized. What is concerned here is more than using mathematics only as a tool which requires to serve certain expectations of the other subjects. For example, project work that surpasses subject boundaries is very suitable for combining subject areas at all grade levels.

Principle 6. *Integration of mathematics teaching with other school subjects presupposes the creation of suitable application models, as well as utilization of information material and methods from other fields of knowledge. On the other hand, we need an understanding of the usefulness and restrictions of the models and calculation methods offered by mathematics. There should not be any curricular obstacles to integration.*

3.3 Evaluation and the use of educational equipment

Evaluation of the student's achievements (or rather their grading) has been traditionally based on the evidence that he has given of his cognitive understanding, when solving familiar, rather short tasks. When in mathematics teaching and in the curriculum a much greater emphasis is now put on the process of learning, the sphere of evaluation should be expanded and its emphasis shifted. We should look for a new type of description of activity and its classification: for example, how rationally the student planned and carried out his work, how actively he influenced the work of the group, what kinds of questions he asked

concerning the problem, and how he assembled and presented the results to the others. By means of such process evaluation the student also becomes aware of his own learning and of the significance of mathematics for his own development.

Principle 7. *The evaluation of learning and the rationale used for it strongly direct the emphasis on the different parts of teaching. When the focus of evaluation is transferred to the learning processes, it is necessary to develop some quite new measurement and assessment methods and ways of giving feedback to the students.*

In the curriculum plan consideration should be given to the use of various helpful appliances, as natural tools in different contexts, as well in learning as in teaching. It is important that the students are given a realistic picture of the importance of computers, in particular, in society and also as tools for an individual person.

Principle 8. *The foundations of the curriculum and other instructions should be formulated so that they make possible the flexible adoption of new appliances and new knowledge in teaching work. Since it is not possible, because of the rapid development of technical devices (e.g., calculators and computers), to estimate all possible future uses and their consequences, the curriculum guidelines must contain the necessary room for flexibility.*

3.4 Options

The leading idea of the comprehensive school mathematics curriculum has been to provide eligibility for further studies, which only too often has meant eligibility for the upper secondary school. This fact has been in conflict with the general educative role of the comprehensive school, and has caused great difficulties for the organization of teaching, both in theory and in practice. Strong emphasis on common objectives for all pupils has often led to a situation where teaching has not corresponded to the abilities of the slow learners or to the challenges offered by the high-achievers.

When reforming mathematics teaching, it is very important to take into account students' co-operative working methods. In principle, each student must have in the learning situations equal possibilities of acquiring and assimilating knowledge, according to his own capabilities, irrespective of his earlier knowledge or sex. An individual school and individual students should have enough possibilities to influence also the content of mathematics teaching on the upper

stage of the comprehensive school and in the upper secondary school. Depending on the special features of the region, opportunities offered by the school and students' interests, it should be possible to introduce optional or alternative content units into the curriculum.

Principle 9. *The fundamentals of the curriculum must provide possibilities for options to the schools and municipalities. The municipal and school curricula should, in turn, offer enough alternatives also to the student.*

4. Implementation of the curriculum change: problems and possibilities

Although the Development Group of Mathematics Education was quite unanimous, when it presented the above-described principles for the development of teaching, it was at the same time very conscious of the fact that their transfer into the practice of mathematics teaching is not a simple and unproblematic task. The group recognized that we must start from the prevailing teaching practices, and therefore it is necessary to find such strategies for the development and renewal that would make the teachers willing to support it. For example, Burkhardt et al. (1989) have stated that "there is no established method of general, planned change of either the implemented or the achieved curriculum", and further, "that planned curriculum change is very difficult to achieve on a large scale". In addition, the curriculum processes of different countries indicate that the implementation of a change is very slow: In England the introduction of the problem-centered teaching practice took about ten years, and in the Netherlands they have reserved seven years for the implementation of a mathematics curriculum reform which concerns 12-16 year-old students.

4.1 Problems in the implementation of change

Many researchers have investigated questions connected with both educational innovations and the implementation of curriculum reforms (e.g. Burkhardt et al. 1989, Fullan 1982, Popkewitz 1988, Wojciechowska 1989). Among other things, they have sought to identify factors that have influenced the failure of the reforms, investigated the prerequisites of success, and analyzed different approaches to the realization of change.

Wojciechowska (1989) and Popkewitz (1988) refer to the following factors, among others, that have had an influence on the failure of curricular reforms in

mathematics: 1) a 'revolutionary' perspective, 2) neglect of the social and cultural ramifications of the reform, 3) concentration of the reform measures mainly on the school level and 4) insufficient consideration and utilization of the public opinion.

The best known example of the revolutionary approach is probably the New Math. As a result of an unsuccessful adoption of its basically good objectives, serious problems emerged in teaching. The second factor mentioned by the researchers emphasizes that the change of mathematics teaching must reach the social and cultural structure of the school community. One of the most essential questions is then, what level of a 'phenomenon' mathematics teaching is seen to be. Most often mathematics is regarded as a static and unified knowledge structure, which has been invented long ago, and whose teaching involves the transfer of this knowledge structure to a new generation. Less often, by contrast, mathematics is understood as a dynamic, continuously expanding cultural product, whose study involves a (social) research process which rediscovers the results of mathematics (Ernest 1989). The conceptions of administrators and decision-makers and the opinions of the teaching personnel, as well as those of the general public, which are born and shaped as a result of many different influences, are equally at issue here. It is often believed that public opinion has influence only on the decision-making level, although it can also exert considerable influence on teachers (Popkewitz 1988).

The third factor that has influenced the failure of the reforms emphasizes the one-sided focus of the reform measures. Usually the reform measures have been focused on the educational system level, and only little attention has been paid to the processes at work within the system, or to what actually takes place at the school and classroom level. The result has been a considerable gap between the achieved curriculum and the original idea. The risk for the emergence of such a gap is greater in a centralized system, where the information dissemination, necessary training and administration of the reform are all in the same hands.

Burkhardt et al. (1989) examine curriculum change as a dynamic system and analyze change processes by means of curriculum research. As a basis for their examination they distinguish four levels in the research and development work: 1) the level of *learning*, 2) the level of *teaching possibilities*, 3) the level of *realizable teaching* and 4) the level of *curriculum change on a large scale*. In analyzing the curriculum change on a large scale (level 4), Burkhardt et al. use the concept of *organizational readiness*, which embraces eight factors that influence the realization of change: ability, values, information, circumstances, timing, obligation, resistance and yields. It would certainly be useful to examine all these factors more carefully, but in the context of the discussion of the Finnish situation at least the questions related to the *resources* (of schools) and the *timing* of the change are central.

Concurrently with the mathematics curriculum reform, the whole curriculum system is undergoing innovation in Finland. As in Sweden, this means that we are moving away from a centralized system towards a more decentralized one; in educational administration one speaks of a transfer from norm-based regulation to information- and outcome-based control. Instead of the earlier practice, where schools followed the normative national curriculum, the National Board of Education now issues the fundamentals of the curriculum plan, and the Ministry of Education confirms the quota of lessons, on which basis the schools then construct their own curriculum plans. The fundamentals of the curriculum plan define, among other things, the responsibilities of schooling, the goals of education, the foundations of the formation of degrees/examinations and the objectives and extent of study units. Another important change is the fact that inspection and approval of the teaching materials by the National Board of Education are no longer required. This way schools are given more freedom and responsibility in their own curriculum work.

Burkhardt et al. (1989) argue that several simultaneously occurring changes make the initiation and progress of a curriculum reform more difficult. A very probable, and perhaps the only, activity pattern for many schools in this situation is that they construct their own 'curriculum document' by slightly changing and modifying the report of the national blueprint. The given task is thus realized at the administrative level, but the more profound process aimed at the innovation of teaching gets less attention, or is even totally neglected. This is what happened in Finland at the turn of the 70s, when the New Math was introduced into the comprehensive school curriculum. At that time teachers were simultaneously faced by the tasks of teaching renewed mathematics and by the practical implementation of the new school system. As a result, mathematics teaching drifted in a very superficial direction, which stressed the symbolics of the set theory. This is not, of course, the only reason for the problems of the New Math phase, but the simultaneous occurrence of the other big change certainly had an important effect (Kupari 1989).

From the point of view of schools and teachers, the above-described procedure of curriculum design is rational and very understandable. They are not easily willing to take up demanding curriculum work, because they are not quite clear why this work is done and whether there are resources available for the work. How is the necessary time for the work going to be found? What kind of training would be needed and where is it given? These are important questions, when we are considering to initiate both school-specific and subject-specific curriculum work. They are also difficult questions, because due to the strong financial saving measures of the state, schools have had to cut down their expenses already many

times. The economic difficulties thus contribute to the weakening of the motivation of schools and teachers for curriculum planning.

4.2 Possibilities of change

The change of the mathematics curriculum is aimed at the innovation of teaching in the way delineated by the above-described development principles. It is therefore necessary to consider, what kind of school-specific curriculum work would most strongly promote this aim. As I said before, it is possible to carry out the construction of school-specific and subject-specific curricula formally, so that the objectives and learning contents prescribed by the national blueprint are placed in the different school years, and the produced result is then entered in the records. This fulfills the administrative obligation set for the schools, but in practice it is likely to mean that the prevailing teaching tradition continues.

One possibility would be to think of the curriculum as a continuously developing *process*. Then the construction of the curriculum plan would form the initial stage before the implementation of the curriculum, and the undertaken work would serve the planning of practical teaching work, as far as possible. The school's curriculum work would thus be clearly aimed at the development and facilitation of the teacher's own work. This work process would be constructed of different development projects, to which all teachers could commit themselves. The development targets might include areas where change is desired or which are in some other way interesting from the school's point of view. For example, a development target could be some content unit, whose study could be planned in a way that departs from what has been done earlier. Or it could involve the utilization of computer technology in teaching, or the combination of technology, mathematics and the natural sciences, etc. The resulting solutions, materials and assessment methods could be immediately tested in practice and, on the basis of the feedback received from the participants, the advancement of the process could be planned. Another desirable viewpoint in this kind of work is that value questions and universal goals of education are not left at the general level, instead, in group discussions solutions would be created that help to direct the teaching situations. The advancement of the process and the most central objectives and policies of the different projects could be described in the written curriculum of the school.

The problematics of effecting a curriculum change were also discussed at the ICME-7 Congress in Quebec in autumn 1992. As a result of the discussions in one group there came up, for example, the following elements that inaugurate and

promote change:

* Teachers are the most important change agents. It is necessary to consider carefully, how to convince teachers of the need for the change, and what support they need during the change process.

* The change must be marketed to the teachers. We must start from selling the change need, not from criticizing anyone.

* The teachers should be allowed to feel that they are an essential part of the change, and that they can influence affairs.

* The teachers should be made to understand that it is always demanding to participate in a change process. As the process advances, participation usually becomes more interesting and perhaps also easier.

* The realization of a change should be started from something that one knows to work, because this increases confidence. The activity is most fruitfully started by those who volunteer for it and are enthusiastic.

* Factors that enhance and help to maintain the change:
  1) functional working conditions
  - open, encouraging and reliable atmosphere
  - enough time for study and for interaction with colleagues (increase of information, exchange of successes and failures)
  - teacher networks
  - in-service training
  2) support from the community surrounding the school and from the parents
  3) appropriate economic incentives

Popkewitz (1988) calls the above-described process approach an *evolution model*, when he presents different viewpoints of change. In the evolution model the change consists of slow and steady movements, which change the social system (the school organization, as well as the studying process). The aim is to try to effect a change in the functional interdependence relationships of the system. The prevailing situation forms the starting point, while the goal of the change is to help teachers to develop such working conditions that combine the new thinking with the old structures. The teachers must be made to understand and accept the

necessity of the change and those assisting measures by whose support they gradually advance towards the desired state of teaching. At the local level the teaching staff discusses the questions that emerge and realizes and directs the change strategies, supported by in-service training. The assumption is that the inclusion of the development philosophy will regulate the existing activity, making it healthier and more progressive.

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The Conceptions of Mathematics Teaching and Learning Are Changing - Is School Teaching Changing?

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School teaching has been criticized in recent times with good reason and to an ever-increasing degree, because not enough attention is given to developing the pupil's thinking and his skills of information acquisition and processing. Instead of just distributing information and teaching ready-made models of solution, the teaching-learning processes should be made more pupil and problem centred. Using empirical data both from recent history and from present times, the current article describes the practical meaning of the implementation of this change and which problems and restrictions will inevitably be involved in making the change come true. Mathematics teaching and learning, with a remarkable amount of systematic knowledge available of historical changes and the present-day situation, is used as an example.

1. Introduction

Mathematics education like education in general is by its very nature an activity with a purpose, and its goals and methodological solutions are dependent both on the scientific and technical development of society and on the economic and physical resources provided by society. This is particularly obvious at the present time with the development in the different disciplines of science and in information technology posing remarkable pressures on the school teaching of mathematics and adding new possibilities to develop it further.

From the viewpoint of preserving and developing the functions of the postindustrial society of the modern times, it is not enough to provide the pupils with the basic everyman's skills in mathematics. What we need is citizens who are autonomous and capable of co-operation, with a more comprehensive all-round education in mathematics and natural sciences, and the capacity and desire to get beyond the immediate prevailing situation and to re-evaluate continuously their own action. Therefore, changes are required in the prevailing teaching strategies which has also been emphasized in recent articles and publications on mathematics teaching and learning. In brief, mathematics teaching and learning should involve more discussion and it should make better use of the pupils' previous learning

experiences and their models of thinking and action (e.g. Bauersfeld 1988, Cobb et al. 1992, Halinen et al. 1991, Lehtinen et al. 1989, Murray 1992, van Oers 1992, Voigt 1992).

It is my purpose in this article to analyze what the implementation of this change means to and requires from the writers of curricula and learning materials and from the teachers in the field. I will also present some of the problems and restrictions which will inevitably emerge when attempts are made to change the modern formalistic and teacher-centred teaching tradition in a more pupil and problem centred direction. As today's situation is, so far as I can see, solidly based on the earlier reform in mathematics education in the 1960s, I will start my analysis with a brief review of recent history, proceeding through an analysis of the current situation to the possibilities and restrictions that will be apparent at the various stages of the reform that we are aiming at now.

2. On the reform of mathematics teaching in the 1960s: Background to the current situation and needs for change

Lots of other things happened in the 1960s besides the radicalization of the student movement. Efforts were made to bring mathematics and its contents in particular up to date. All of this took place as part of a more general atmosphere and social change, in which the transfer of the newest trends and results in various disciplines of science to teaching was emphasized. Conveying the general basic structures discovered in the different sciences to the rising generation became the most important goal. In accordance with the general trend, the existing school system which "adapted the pupils to life" was criticized and it was demanded that the structuralistic nature of modern mathematics should show more clearly both in the universities and in schools.

The change was initiated by a number of top figures in theoretical mathematics, headed by Jean Dieudonne, one of the founders of the Bourbaki group (Fehr 1961). Although critical opinions were already heard when the efforts for change were only beginning (e.g. Ahlfors et al. 1962), they could not prevent the ideas focusing on the formal structures of mathematics from being absorbed into mathematics education right from the university level to primary education. Therefore, mathematics syllabi and learning materials were changed radically in the 1960s and early 70s in many western countries, including Finland. For instance, the verbal real-world problems typical of mathematics in the senior primary school of the 1950s had to give way to the notations of the naive set theory and formal logic.

It was typical, for instance, that the set of natural numbers was discussed when natural numbers were studied, and teaching them was started from various classifications and examinations of one-to-one equivalencies. The number field was then extended to the rational numbers and real numbers. The reductionist efforts in the early 20th century to "base" mathematics on natural numbers and formal logic and the ideas of mathematics as a reserve of information on axiomatic formalistic structures thus found its counterpart - not a valid one though - in school mathematics (Freudenthal 1973, Keranto 1981).

As we know now, this reform was not a success. Some of the teachers, mathematicians and parents of pupils were not willing to "understand" the justification of the changes, and they required in newspapers, for instance, that one should return to a mathematics teaching that has more of a content and is more practically oriented (for a deeper criticism of "modern mathematics", see Freudenthal 1973, Kline 1978, Thom 1973). As there was nothing to boast of in the learning results, either, it was admitted by the National Board of Education in the mid-1970s that the reform had gone too far. It was admitted that examining various sets and descriptions and the notations of the set theory and formal logic had become an end in itself that was contrary to the real purpose. It was admitted that the direction of school mathematics had to be changed in the direction of "everyman's mathematics" with a more practical orientation. This is also what finally happened, but not until new curricula and learning materials were developed in the 1980s (Kouluhallitus 1982, 1985).

3. What should we learn from the reform in the 1960s?

First of all, the setbacks which the reform met with seem to be a good example of what happens when the basis for changes in the curriculum and learning materials is provided by a one-sided view of the nature of mathematics and inadequate knowledge of the significant and meaningful teaching/learning processes of different mathematical contents.

There is much more to mathematics than internal calculation of readily given uninterpreted structures or deduction from given axioms. The basis of mathematics lies in the solution of quantitative, spatial and temporal real-world problems. This explains the usability of the general mathematical structures which are independent of the context in the solution of many practical life and scientific problems (e.g. Freudenthal 1983, Maki & Thompson 1973, Nevanlinna 1971). Furthermore, if mathematics is examined from the viewpoint of research processes, it turns out to be so that mathematical activity is both an individual process of construction and

a community-bound social activity. In other words, both creative problem-solving processes and discussions and negotiations of the consistency and validity of the solutions made are needed in the development of mathematical systems. The process of mathematical development involves counterarguments, criticism of earlier research and examination of facts from new viewpoints (e.g. Ernest 1991, Lakatos 1976). Therefore, when modern curricula and learning materials are developed, both the structural side of mathematics and the creative side based on disputation and adaptation should be considered.

As for making use of the research into educational and learning processes in the construction and justification of curricula and learning materials, expertise, responsibility and caution is required from those who make use of such research and those who do the research alike. For instance, the reform described above should not have been justified by Piaget's research. In fact, the research in question was made for purposes other than the development of school instruction. The most important goal was to find the origin of human knowledge formation and the general regularities of the process of intellectual development (genetic epistemology, Piaget 1970). Piaget presented no explicit implications or applications of how the learning of scientific knowledge should be arranged, although the research results can be thought of as supporting different pedagogies based on the pupils' own action and invention.

What would have been needed was a more many-sided and detailed knowledge - comprising culture, language, thinking, social processes - of the teaching/learning process of each mathematical content to be learned and of the strengths and weaknesses connected with each teaching strategy. If the current knowledge of arranging meaningful learning processes had been available, then mathematical formalism would hardly have occupied such an important position in curricula and learning materials that it actually managed to grasp.

Secondly, the difficulties which have been observed clearly indicate that any reform needs the broad support of those who are responsible for mathematics education and those who develop it. In other words, the existing teaching tradition and social practice need to be taken into account and the changes need to have a realistic basis. In the reform outlined above, all the ties with the previous practically oriented mathematics teaching in schools were broken, and the requirements made by practice and working life on changes in mathematics syllabi were not sufficiently taken into account. It was, therefore, difficult to convince the parents and some of the teachers - and it is still the same - why everyman's mathematics with its mental calculation and written real-world related problems had to give way completely to exercises involving the set theory and mechanical calculation.

4. Present-day situation: Do we need a change and if we do, then what kind of change?

The changes in the mathematics syllabi and learning materials discussed above are still to be seen in today's learning materials and curricula as well. In accordance with the behavioristic tradition, people still speak about the acquisition of this and that mathematical conception and about the essential attributes of the concepts, and people still plan their teaching on the basis of small and short-sighted learning tasks. It is still thought that the teaching of many mathematical skills and items of knowledge, both on the primary and secondary levels, is to be implemented in terms of the formalist approach - "first theory, then the applications". A good example of this is the teaching of ratio and proportionality. Although there would be excellent possibilities to base the teaching on real-world problems and discussions about their solutions, the curricula talk about the concepts of ratio and proportionality and textbooks proceed according to the strategy referred to above: first the important terms and definitions, then practising the algorithms and finally some practical exercises (for more details, see Keranto 1990, 1991).

Although this kind of teaching strategy is relatively easy to plan and "safe" to implement for the teacher and the authors of learning materials, it does not meet with the modern views and requirements for what kind of mathematical skills and how pupils should study and learn. In fact, good theoretical and empirical grounds exist for an assumption that this kind of teacher-centred teaching strategy, based on offering and using ready-made models of solution, will lead in many cases to rote learning and lack of experiences of meaningfulness. This is the very point where some change is needed. The teaching/learning processes should be more problemcentred and they should make better use of the pupils' existing real-world experiences and informal models of thinking and action. To use Freudenthal's metaphor, "the horse should now be harnessed to the carriage and not vice versa", which has been the "antididactic practice" for quite a long time in mathematics teaching (Freudenthal 1983).

Freudenthal recommended a solution which is inverse to this strategy of "concept acquisition". In this approach the teaching/learning process would be started from those situations and problems in which the mathematics to be learned is really needed and can be made use of. The same ideas can also be found in the research and recommendations of the cultural historical school of Soviet psychology (Davydov 1982, Engeström 1983, Leontyev 1977, Vygotsky 1962, also see Keranto 1990). At the same time as mathematics teaching is linked in a more valid way to the solution of real-world problems, a teaching/learning process involving discovery and discussion becomes possible. This would also be better in keeping

with the general goals and research-based ideas of learning which have been recently presented of teaching and learning in general and of mathematics teaching and learning in particular (for the constructivist perspective on teaching and learning, see Cobb 1990, Cobb et al. 1992, Lehtinen 1984, Murray 1992, Wheatley 1992, cf. conception of learning, Halinen et al. 1991, Lehtinen et al. 1989).

5. Needs for change and practical teaching: Problems and restrictions

It is not so simple to exchange the role of an information distributor to the role of someone who initiates, directs and moderates discussions. The author has experiences of this both personally in teaching and as an outside observer. Let us, for instance, examine the following episode which was recorded in an experimental class at the beginning of the third lesson (see the 4th teaching experiment of the skills of proportional reasoning and calculation, Keranto 1992, also see Kaartinen 1992). The lesson was started with the home assignment detailed below. It should be mentioned here that different kinds of liquid mixture problems had been solved and ratio had been introduced in the previous lessons.

"A fence needs to be painted. Painting each four metres of fence takes 2 litres of white paint and 8 litres of red paint. You already have 8 litres of white paint in the storeroom."

(a) How many litres of red paint do you have to buy?

(b) How many metres of fence can you paint with the mixture to be made?"

The joint examination of this problem proceeded as follows (t=teacher; p=pupil; comments by author):

t: "How many litres of red paint do you need?"

p(1): "14 litres"

t: "How did you calculate that?"

p(1): "I used subtraction" (a typical erroneous strategy)

t: "p(2)" (teacher neither corrects the previous solution nor continues from it; why?)

p(2): 32 litres

t: "How did you calculate it?" (teacher is still using the term 'calculate')

p(2): "There is four times more red paint than there is white paint. So I multiplied the amount of white paint by four. " (p(2) is using a faulty expression, mathematically; he should say "four times as much as"; note the mixed usage even in the media!)

t: "Are there any other solutions?" (teacher does not initiate a discussion of the conflicting expression)

p(3): "There's four times the amount of white paint here" (p(3) has compared the amounts of white paint to each other: further specification would be needed)

t: "There's four times the amount of white paint in here. So we also have four times the amount of red paint." (teacher specifies the answer provided by p(3) but does not ask him to do it himself)

t: "How many metres of fence can be painted?"

p(4): "16 metres"

t: "A fourfold amount" (teacher does not inquire about the solution process and is content with giving himself a vague hint of the process) t: "How many of you did get the right answer?"

This was a highly typical episode. The teacher relied on old concepts and his experience, although he would have clearly liked to use a teaching strategy involving more discussion in accordance with the ideas and instructions presented in the teaching package. It is likely that the next time he would have done better, providing that the teacher finds out in advance how the pupils will be solving the problems and how different solutions are utilized at the various stages of the teaching session.

6. An example of the pupil and problem centred teaching tradition

A good example of the fact that mathematics teaching can be implemented in a pupil and problem centred way even on a larger scale is provided by Stigler et al. (1992). He compared in great detail the teaching practices in elementary schools in the United States and Japan. The idea was that any differences in the planning and implementation of lessons might account for the good success of Japanese pupils in international comparisons. The most important goal was to find out what kind of chances for discussion the pupils are given in practical classroom situations in both countries and how the teaching strategies differ from each other in other ways. It was discovered that while the pupils in American classes had little opportunity for discussion, those in Japan were given plentiful opportunities. Also, while the Japanese class teachers concentrated on considering and utilizing the pupils' own solutions in their planning and implementation of teaching, the American teachers based their lessons on what the teacher would say and do at each stage of the lesson.

The Japanese lessons which were observed could be divided into four segments as follows:

(1) Introduction of the problem (e.g., determining the areas of different triangles) (some 5 % of the lesson)
(2) Autonomous efforts by the pupils to provide a solution (some 30 % of the lesson; the teacher circulates and provides diagnoses)
(3) Classroom discussion of the pupils' solutions and finding a general model or pattern of solution (some 50 to 60 % of the lesson; the pupils describe and dispute their solutions; the teacher's questions are meant to clarify their thinking; solutions which are weak or invalid in objective terms are also discussed)
(4) Pupils doing further exercises in the textbook on their own (some 5 to 15 % of the lessons)

The teachers included four types of information in each stage in their plans: (1) learning activities; (2) expected pupil reactions to the activities; (3) time to be presumably taken by each stage; (4) guidance and advice.

As we can see, the lessons of the observed Japanese elementary school teachers can well be described as pupil and problem centred, while the lessons of the observed American teachers can be described as teacher-centred and as ones which offer ready-made models or force pupils to use them. This was the case in spite of the fact that there has been a lot of talk in the United States for quite a long time about changing mathematics teaching in a more pupil and problem centred direction.

So far as the American teachers are concerned, the results of the above survey are not surprising at all. Although the teachers would like to and/or thought they did act in accordance with the new constructivist ideas, the old models of thinking and action connected with teacher-centred and formalist teaching strategies were still determining the course of their lessons (see e.g. Darling-Hammond 1990, Sykes 1990). In other words, as I said above, it is by no means a simple matter to exchange the role of a distributor of information to the role of one who initiates, directs and moderates discussions.

7. Conclusions

It becomes apparent from what I said above that the prevailing teacher-centred teaching tradition which is based on imposing ready-made models of solution does not provide a sufficient foundation for achieving a many-sided mathematical education. Changes are needed in the views both on mathematics and on teaching and learning it.

It is justifiable and fruitful to understand mathematics learning both as an individual construction process and as a process of enculturation into the meanings and social practices of the society in a broader sense (e.g. Bauersfeld 1988, Bauersfeld et al. 1988, Cobb & Steffe 1983, Cobb et al. 1992, van Oers 1992, Voigt 1992). The pupil interprets and constructs the relations between his own knowledge and the new knowledge offered to him in terms of his own previous experiences and goals. Therefore, the things to be learned should also have some personal significance to the pupil in addition to their links with the pupil's previous experience and knowledge (cf. Engeström 1983, Leontyev 1977, meaningful learning). In addition to the individual constructivist side, mathematical activity and therefore learning it essentially includes the social perspective involving agreement on the joint courses of action, the validity of conclusions and the meanings of the language of mathematics (Davydov 1982, van Oers 1992, Vygotsky 1962, cultural meaning of things to be learned). In other words, only joint discussion and other available information will reveal what the pupils' idea of the relevant questions is, how well it has developed and where there still are shortcomings and points needing correction. Such discussions on joint courses of action and solution processes direct the pupils to adjust their own language and thinking which is of vital importance to an understanding and transferable learning of mathematics (cf. Galperin 1957, phased internalization of external acts, Wheatley 1992, reflection in mathematics learning).

Of course, it is not enough just to accept these ideas and record them in, for instance, mathematics curricula. First of all, changes need to be made to the system of lessons. The current 45-minute system is much too rigid. It seems to constrain the planning and implementation of prolonged problem solving processes and directs teaching towards the provision of ready-made models of solution. Secondly, learning materials need to be designed according to the new ideas of teaching and learning and they need to be tested critically in ordinary classrooms. I find this to be a lengthy and demanding process with no shortcuts, similarly to mathematics in itself. In fact, teaching based on constructivist ideas means, first of all, that the pupils are given good opportunities for autonomous thinking and discovery right from the start of the teaching/learning process, just like the teachers in Japanese

elementary schools seem to do according to Stigler. It also means that enough time is reserved for discussions about possible solutions and that it is ensured that the pupils have a good opportunity for constructive criticism, explanations and defence of their own solutions. Thirdly, it also means that, when planning each lesson, the teacher will already try to consider how different problems will be solved and how the pupils' own solutions (valid and invalid ones) will be taken into account in the teaching situation. Fourthly, it means that the teacher follows the pupils' actions when the problems are being solved and develops methods of assessment for the follow-up of the learning process.

I think that only then we can speak about pupil and problem centred teaching which will hopefully gain ground gradually from teaching strategies representing "concept acquisition". Only then the pupil has real possibilities to acquire a many-sided and realistic picture of the nature of mathematical activity and of the basis and uses of mathematical knowledge. In addition to this, there will be an increase in the possibilities of an understanding and meaningfully experienced learning process and thereby widely transferable mathematical skills. The time has come for some real action.

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Organizing Ideas from the History of Mathematics Drawn from the History

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The problem of "math dropouts", always a serious one, has increased now that "mathematics for all" has come into fashion as a slogan. The subject matter of mathematics is normally presented as a meaningless collection of knowledge - unrelated to the experience of the students and totally uninteresting. They are presented with sterile "logical connections" which they seldom understand or appreciate and as a result, they are very prone to error! This has given rise to a flourishing enterprise - empirical research - which studies and characterizes the symptoms without producing a cure. We need a new approach to the teaching of mathematics but there is little hope it will emanate from this psychological perspective. Epistemological perspectives and historical sources offer a much greater hope. Besides, they must not be forgotten when planning curriculum or constructivist learning environments for pupils' productive activity.

1. Introduction

The history of mathematics abounds with outstanding examples of simple, but powerful ideas for organizing the content of the curriculum in a meaningful way. Traditionally, the same idea is treated in several disguised forms under the guise of "spiral curriculum". Thus, for example, *similarity* governs the equality of fractions, $y = cx$, the classification of triangles etc. Do students see how these seemingly disparate ideas can be related to one unifying idea or do they simply memorize a lot of facts (like twenty theorems in geometry)? Similarly the theorem of Pythagoras is closely related to the equation of the circle and to simple properties of the trigonometric functions yet their separation for instance in the German or Finnish curriculum is measured in years.

In contrast we suggest firstly the liberal use of Ockham's razor, then the connecting of simple but powerful ideas within a simpler network. In this way we are led to a position, from which Pascal's "esprit de finesse" (from a few central points, short paths can be laid to many stations!) will more greatly rule the process of thinking and by extension, the process of learning.

Some impressive examples will show that the less able students especially will profit from such an organization of mathematical knowledge. Following some general comments on "organizing ideas" we present two paradigms:

- A problem solving approach to an idea of Pascal (for organizing the divisibility rules found in Arabic writing) suitable for 11-year-olds
- A proposal of Leibniz to base geometry on distance and symmetry adapted to the classroom.

2. Organizing ideas

Insight, understanding, comprehension are all metaphors reflecting the perception of the primary relationship of the various parts to the whole. Our first view of a person, or a landscape from a mountain top gives us a "total meaning", which is simpler and more easily retained than the multitude of individual "meanings" of the constituent parts. The total meaning (the idea, the plan) provides an ongoing clue to the parts, and how the parts connect together into an integrated framework.

Mathematical "objects" (numbers, similarity, functions, fields, ...) are infinitely more simple than people or landscapes, and are not "learned" in the same way, nor are they seen with real but with Platonic eyes. There are some mathematical concepts for which it is appropriate firstly to learn the "parts". However, the usual practice is always to direct teaching to these individual parts without any real view of the "whole" - the networking of these parts. This "atomized" mathematics, much easier for unimaginative teachers and perhaps more suited to "learning for the test", is seemingly justified by referring to psychological principles such as the principle of "isolation of difficulties". The problem is that this approach reduces the subject to a collection of isolated facts, leaves the student completely unmotivated and leads to error producing behaviors with only reproductive activity (Haapasalo 1985). This no doubt accounts for the lack of success in the teaching of mathematics to all students.

Attempts to solve the problem - after the fact - such as error correcting, remediation, overcoming "math anxiety" provide great opportunities for psychological "research" but do little to repair the damage.

There is a great deal of lip-service and well-intentioned advice given as to the use of history in the teaching of mathematics, but there are very few worked out examples of the kind we would like to discuss. In the history of mathematics we were on the look-out for ideas ("organizing ideas" that were
- influential in the development of mathematics
- simple, useful and powerful which at the same time could act as
-"centers of gravity" within the curriculum
- nodes in a cognitive network.

Often, in the course of history, new central ideas developed out of the reorganization of the old stock of knowledge leading to a better general map, a "higher point of view". "La vue synoptique" brings to light associations hitherto hidden.

This does not work by "longues chaînes de raison" (Descartes). Nor does Euclid's "l'esprit de géométrie" rule the process of cognition and by analogy the process of learning, but rather it is governed *by a mode of thinking related to Pascal's "esprit de finesse" - to lay out short paths from a few central points to many stations.* *)

Pascal himself provides a splendid example. When sixteen years of age he reorganized the knowledge of conics handed over by Desargues unveiling

the "mysterium hexagrammicum" (ever since called Pascal's theorem): the high point surrounded by a lot of close corollaries.

Having cut organizing ideas out of the historical context one is left with the hard task to process them into more or less comprehensive teaching modules (or even schoolbook chapters). The products can be problem fields in which, instead of dozens of theorems, very few organizing ideas operate as means for problem-solving. Concentrating on few simple butpowerful ideas which organize their environments and which are connected one to another within a simple network, offers help for the less able student as well. His lack of success derives to a large extent from the fact that he is unable to organize his thought with respect to a complex field in which the connections are presented in the usual plain logical systematic way and where teaching is used to administer only spoonfuls of the subject matter, disconnected and without depth.

Ransacking history, pasting together ideas, looking for interesting problems of very different provenance, even some routine exercises might not always please the feelings of the "real" historians. Junior high school teachers, however, should find the "real bricks" for interesting problem-oriented teaching modules in the following examples.

3. A textbook chapter based on an idea of Pascal

Suppose we have to teach some divisibility rules. (e. g. the German syllabus requires teaching the rules for a few divisors 2, 4, 8, 5, 9). When Pascal was living in a monastery near Paris he was confronted with the same pedagogical task. He thought about the problem, and wrote a not too wellknown paper about a general digit-sum-rule.

Generalizing Pascal's approach we will develop in line with his idea, a quick algorithm for remainders. The organizing idea behind it provides far more than merely a real understanding of all the special cases (divisibility by 2, 4, 8, 5, 9).

How can students discover this organizing idea? Pascal does not answer this pedagogical question. He wrote the paper as a mathematician: explaining the algorithm by some examples (9-, 7- rule) and then proving it by induction. His approach is certainly not appropriate for an interesting lesson on divisibility rules for 11-year-olds.

Take an approach with the very familiar clock on the face of which, so to say, Pascal's idea, and even the more general idea of congruence comes to light in a very simple way. We quote from Stowasser (1982):

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*) "... il faut tout d'un coup voir la chose d'un seul regard, et non par progres du raisonnement, au moins jusqu'à un certain degré. ..." (Pascal, Pensées, éd. Lafuma 512: the difference between l'esprit de géométrie and l'esprit de finesse)

In the 11-year-olds' daily life experience the clock is just the right thing to start with. I have a big cardboard clock without a minute hand.
The hour hand is on the 12.

A pupil is called to the blackboard. He is told to write down the number of hours the hour hand should move. He writes down an unpronounceable number of hours which goes from left to right across the blackboard. Three seconds after the last written figure I put the hour hand on the right hour. For example, the pupil writes

20450102230531234567890246813579025414 03.

I put the hour hand on the 7. The pupils check by dividing through by 12. One page is filled. Plenty of mistakes. In the right hand corner is written the only interesting thing: the remainder.

The pupils know that I am not a magician, especially that I am not good at mental arithmetic. Of course I do not reveal the trick. The pupils will work for it, discover it. A prepared work sheet asking "what time is it", that means asking for the remainders regarding 12, shows a pattern: the remainders of 12 divided into the powers of 10 from 100 on, are all equal, thank God.

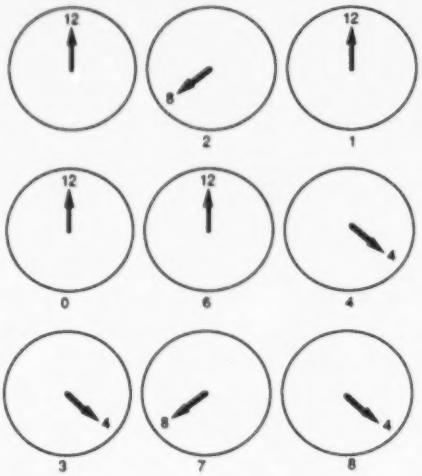
$$R_{12}(10^n) \text{ is constant for } n \geq 2$$

Every 10th power pushes the hour hand 4 hours ahead.

I assume, otherwise it has to be dealt with, the pupils really know what the abacus is, that they can see a decimal number consisting of powers of 10.

Now my mystery trick in arithmetic is solved. No matter how many digits there are in front, the hour hand simply jumps to and fro among three positions (beyond the tens).

I prefer to do the rotation of the hand mentally instead of on the actual clock. In the example $R_{12}(2106437822)$ my mental arithmetic looks like this:



The 22 hours at the end, being out of the routine, put the hour hand in the final position:

The 11-year-olds even understand my enquiry whether the calculations on the working sheet confirm that R(10000000000)=4. The reason why - hidden reasoning by induction because of the recursively defined powers of 10 - can be found by 11-year-olds with a little help.

4 hours remain from 10^2 hours after taking away the half days. From $10^3 = 10 \times 10^2$ hours remain 10×4 hours. From 10^3 hours remain therefore again 4 hours after taking away the half days. We proceed in the same way for $10^4 = 10 \times 10^3$.

No doubt, some pupils saw only the repetition in the division scheme:

```
          8333...
      ┌──────────
12    │ 1000000000
        40
         40
          .
           .
            .
             4
```

Howsoever the ability of 11-year-olds to deal with loops in programming their microcomputers, disproves the old didactical prejudice that even much older students cannot understand recursion.

Could we think that our 11-year-olds can now handle a non-babylonic clock?

Of course, and not only the 11-clock, but a 37-clock, if necessary. The 2-, 4-, 8-, 5-, and 9-clocks are really trivial cases.

What of the divisibility rule for 11 and the rules required by the syllabus? Pupils understand the reduction principle:

$$R\,(2352843) \equiv R\,(2-3+5-2+8-4+3) \equiv R\,(9) \equiv 9.$$

Therefore 2352843 is not a multiple of 11.

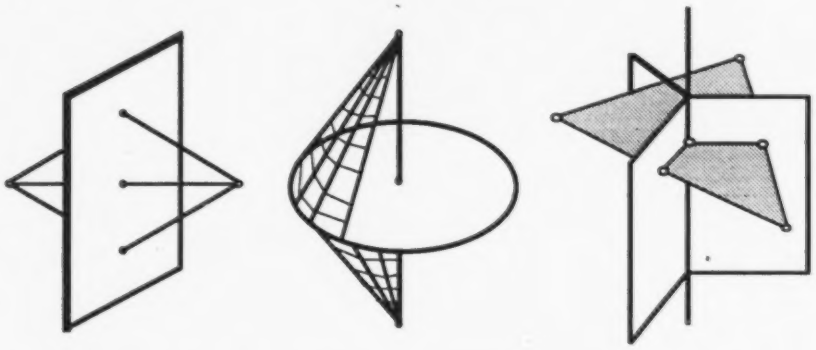
The pupils have grasped *the idea of congruence* which (together with the idea of recursion implemented in the decimal system) completely organizes the mathematics around the divisibility rules (and much more, as a look into the history of number theory shows; Stowasser 1985).

We take this opportunity to attack another psychological principle. Contrary to some popular prejudices to "teach" a mathematical idea it is not necessary to burden the students with a lot of examples from which the idea is to be abstracted. It is far more effective for the teacher to expose the idea by an appropriate paradigm such as, in our case, the babylonic clock. Having grasped the idea from that paradigm the student must then apply (transfer) this idea to new situations. Aristotle has already pointed out the difference between research (many examples) and teaching (the best example; paradigm).

4. A proposal of Leibniz to base geometry on distance and symmetry prepared for the 11-13-years-olds *)

The following sketch of materials prepared for the classroom was inspired by a letter of Leibniz to Huygens dated September 8, 1679 (von Engelhardt 1955) . In this letter Leibniz was motivated by a desire to find an algebra (his Characteristica Universalis) by which the truth of philosophical arguments could be determined. Contrary to Descartes who translated geometry into an algebra of numbers, Leibniz proposed an algebra where geometric objects and relations were represented by symbols and manipulated according to certain rules. The proof of a theorem would be reduced to a calculation with these symbols.

He complained that Euclid's foundation of geometry was too cumbersome. For his purpose, Leibniz chose as a fundamental concept *distance* to which he related *symmetry*. All that was needed to specify entities of elementary geometry in space, were points and the distance between them. Thus a *plane* was defined to be set of points equidistant from two fixed points. A *circle* is the set of points at a fixed distance from two points. A *line* is a set of points equidistant from three nonlinear points.



Not only is this approach to geometry fundamentally different to Euclid's foundations of geometry which have dominated the first systematic treatment of geometry in the schools, but the order of development is essentially reversed. Yet it possesses logical rigour, is straightforward and intuitive, and shows the deductivist approach based on Euclid to be unnecessarily complicated and obscure. As such, Leibniz's suggestions to base geometry on the concepts of 'distance' and 'symmetry' are in accord with contemporary view points of school geometry and have promise for introducing such elementary geometric idéas to children.

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*) This is the main part of a jointly written paper; Booker et al. 1984.
For a more comprehensive version, see Mohry et al. 1981

4.1. Introducing the concept to children

As a starting point, it is only necessary to take from this proposal of Leibniz the idea that distance is the fundamental notion of geometry. A problem which can set the scene for such a development could be one of determining a system of 'school zones' so that children attend the school nearest to their home (Booker et al. 1984, Mohry et al. 1981). From a city map like that below, perhaps their own city, children should first identify the schools, marking them in some distinctive manner.

A first level problem would then be to have them locate certain addresses (by reference to coordinates, by locating their own home, etc.) and determine which school people living at such an address should attend. This could be done by using a ruler, a piece of string or even a compass, and the children could be led to see that a set of circles centred on the home could be used to determine the closest school.

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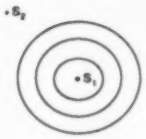
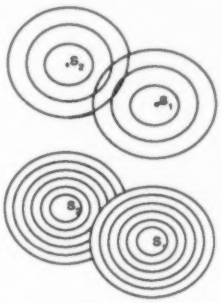
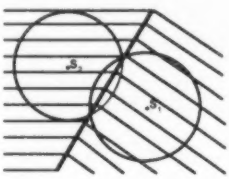
At the next level, the problem becomes one of determining which areas of the city would be assigned to a particular school, so that the focus shifts to circles centred on the schools. The general question, 'How can the map be divided up to show the zones for each school?' can now be posed.



As a first step, the problem must be *simplified* to enable the fundamental nature of the problem to be grasped, so only 2 schools should be considered. Circles may be constructed about the one of the schools (s_1) and questions asked to draw out from the children that beyond some radius, points are closer to one school than the other. Drawing circles of the same radius around the other school as well shows how the zones are forming, and this may be made clearer by using successively finer sets of concentric circles.

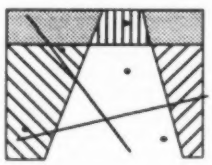


This in turn leads to the idea that the dividing line between the two schools must be a straight line formed by the points of intersection of these circles; that is the perpendicular bisector of the line joining the two schools. This point can be verified by a paper-folding procedure and so two distinct but related methods for determining the dividing lines have been established: by *symmetry*, folding one point onto the other, or by *distance* using circles of the same radius.



An understanding of how the dividing lines are determined can now be applied to all of the schools. It is clear by inspection that the zones must be convex, and mathematical arguments can be used to justify this conclusion at a level appropriate to the level of the children. Consider just one school. Each dividing- line cuts the whole region into two parts, the nearer of which contains those parts in the school zone, the part further away is in another zone. Shade or even cut-off, one at a time, each of the parts not in the zone: a convex region remains.

The problem of the school zones has been solved very simply by an application and interpretation of the idea of a dividing line. At the same time, this application has built up the understanding of the essential nature of symmetry and distance; it has shown the essence of geometry of which Leibniz spoke.

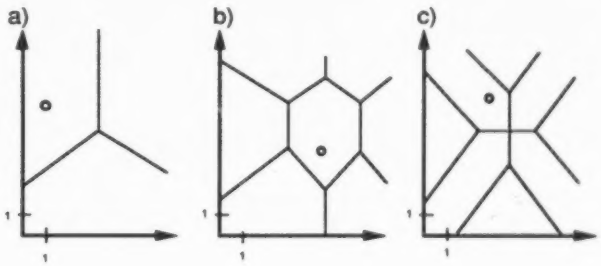


4.2. Generalisation of the method

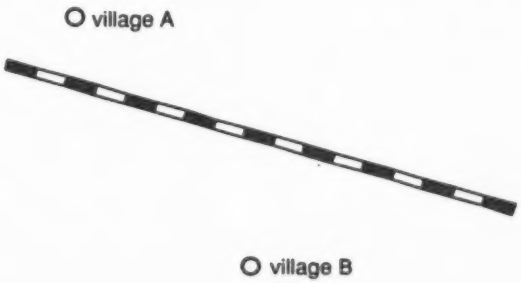
Once the fundamental idea of the dividing line and its relationship to symmetry and distance have been grasped, extensions of this approach can be made using a sequence of problems such as that which follows.

Type 1. Problems which are direct applications of the method

1. On a map or diagram with several points but only some dividing lines, provide the remaining dividing lines and locate the regions.



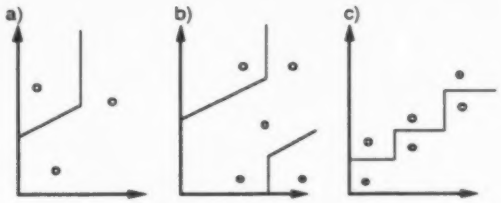
2. Two small villages are adjacent to a straight railway line to the nearest big city. Where should a station be placed so that it is equidistant from each village? What if the railway line is curved, rather than straight?



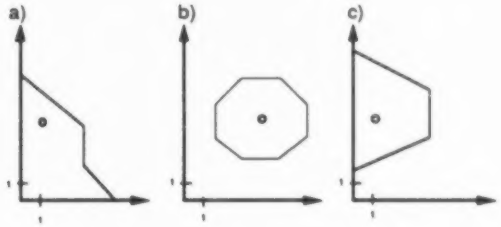
3. Two villages are 5 km apart. A new railway line is to be constructed, but there is no land available closer than 3 km to either village. The station must be equidistant from both and more than 5 km from either village. Can such a position be found?

Type 2. Problems which require the process to be reserved

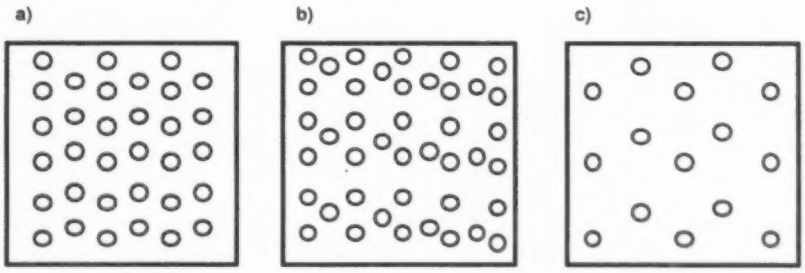
1. On a map or diagram with all of the dividing lines but only 1 point, locate the other points.



2. Given one point and some of the boundary lines, locate the fewest number of points and complete the regions.

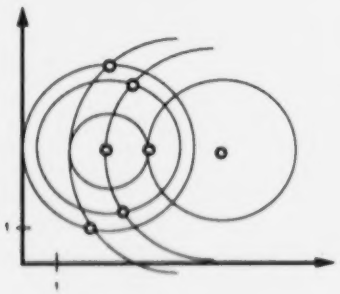


3. Provide copies of uniformly distributed or symmetric patterns of points on which the children could sketch the dividing lines free hand, and then colour the resulting patterns.

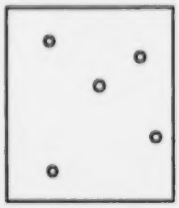


Type 3. Generalisation of the method

1. Consider distributions where the distances from one point to the other are proportional rather than equal. (This gives rise to conic sections as dividing lines rather than straight lines.)



2. In a more realistic situation for school zones, the distances might be measured along roadways rather than simply in a straight line. A hypothetical city in which streets cut at right angles would not be too difficult, but could the original problem also be solved this way?



3. Electoral boundaries for municipal elections are drawn up so that approximately the same number of electors are in each electorate. Could this method be extended to draw up such 'zones'?

Type 4. Old problems made easy

This method adapted from Leibniz can also be used to solve more readily problems that in the past have relied on formal and mechanical applications of Pythagoras Theorem. A similar approach is expected in schools today, so that such problems are usually left until the children have become familiar with Pythagoras result, and occur only as routine applications of it. In 'Chin Chang Suan Sin' (The Nine Books of Arithmetic) published in China around 2000 years ago (Vogel 1968), an interesting collection of such problems were posed and solved in the following manner:

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We take as an example the problem 13 from Book IX:

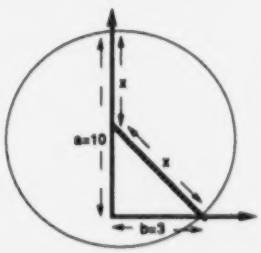
A bamboo stick has a height of 10 feet. The upper portion is bent and falls to touch the ground 3 feet from the foot of the stick. Question: at what height is the bamboo broken?

The following solution was given in the book:

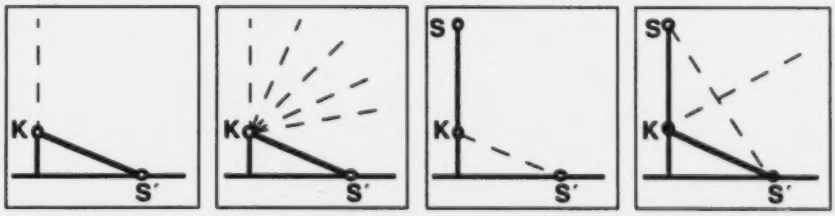
Take the distance to the foot;	3
multiply it by itself;	3 x 3
divide this product by the length.	(3 x 3) : 10
Substract this from the height of the bamboo,	10 - ((3 x 3) : 10)
and halve the difference.	(10 -((3 x 3) : 10)) : 2
Then you have the height of the break	$4\frac{11}{20}$

In a behaviourist school teaching today, pupils have to learn similar routines, although symbolic notations are used in the application of Pythagoras Theorem. What would the anthropologists say after thousands of years about our silly way of using symbolic algebra for simple problems like this!

If the children are familiar with the properties and power of the dividing line process used in the earlier problems, only a small step is needed to quickly and simply solve the new problems. The method is intuitively obvious and does not require Pythagoras' Theorem at all! However, it is not necessarily easy for the child, who must be guided into its use, nor is it so easy for the teacher, who is asked to forego a reliable but mechanical tool. Indeed, it is unusual for a teacher to even consider or encourage such an intuitive approach.



The means to achieving such a transfer has been stressed by Polya (1948). It is to 'work backwards, to begin by assuming the result and then seeking a step or series of steps that will provide the solution. Such an approach of course was simply the 'analytical method' known and used by the Greeks. Returning to the original problem, the following thinking provides the solution (Mohry et al. 1981). Merely using the notion of *symmetry* to find the dividing line fixes the point K, and the height of the break can be measured.



assume the result is already known; let K be the breakpoint

the stick must have turned around K

the distance SK must be the same as S'K

K must lie on the dividing line. Symmetry!

The other problems from this source, and from similar collections from Babylonian tablets of even earlier times, can also be solved using the notion of symmetry far more simply than the solutions that accompany them.

Problem 6 from Book IX: An aquarium has a square base of side 10 feet. A reed grows at the centre of the tank, extending 1 foot above the surface. If the reed is pulled to one side, it just reaches the edge of the aquarium at the water line. Question: What is the depth of the water and what is the length of the reed?

Problem 8 from Book IX: A wall is 10 feet high. A plank leans against the wall so that its height is the same as that of the wall. If the plank is moved so that the end resting on the ground is 1 foot further away from the wall, the plank is just able to rest along the ground. Question: What is the length of the plank?

Problem 10 from Book IX: A double door is ajar, with both doors equally open. The distance from the opening to the doors is 1 foot, and the two doors are 2 inches apart. Question: What is the width of the double door?

5. Conclusion

A way to present important ideas and applications of elementary geometry has been taken from the history of mathematics. Not as an anecdote, nor in its original form, but as a basis for developing a meaningful and motivating approach. Such an approach clearly has much potential for contributing to a rebirth of geometry in the school; to rescue it from the constraints of the Euclidean perspective and to bridge the void which opened between the 'old' ways of Euclid and the 'new' transformational geometry. It has also brought to the surface the type of thinking and use of intuitive ideas that lie at the heart of successful problem solving.

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Special Training of Students for Mathematical Olympiads

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During the last three decades mathematical contests have been proliferating all over the globe. No details about the importance of such competitions in discovering and developing mathematical talents will be repeated here. The following is a brief outline of the history of mathematical contests and the training system for olympiads in Estonia.

1. Historical background

The First Olympiad took place in Greece in 776 B.C. as a competition of deftness, strength and beauty. The Olympic Games were stopped in 394, and this tradition was restored 1896. But two years ealier, in 1894, in Hungary by the initiative of Baron Lorand Eötvös Mathematical Olympiads were started as annual contests of solving mathematical problems for senior high school students. Soon such contests took place in many countries, in Romania, Russia, USA etc.

After World War II a new period in the history of mathematics competitions began. Mathematical Olympiads started in the so-called "socialist countries", but the first International Mathematical Olympiad (IMO) took place in Romania only in 1959. Originally it too was a competition between the socialist countries only. Later the number of the countiries partcipating in IMO (first only from Europe) grew from year to year (for example, Finland took part in IMO in 1965; England, Italy, France and Sweden - in 1967, etc).

At the present time IMO has been turned into a fully international competition. During the last twenty years the number of various mathematics competitions in the world has multiplied greatly. To exchange ideas and information about mathematics competitions in different countries, the World Federation of National Mathematics Competitions (WFNMC) was founded in Australia in 1984. Soon (since 1988) this organization began to publish its journal "Mathematics Competitions". The competitions became more and more popular eweryhere in the world. In October 1989 the UNESCO General Conference on Education resommended that member states should promote out-of-school activities

such as the international and regional olympiads in the sciences and mathematics in order to encourage scientific talent and initiative among young people.

2. Estonian mathematics competitions

In 1950 the Estonian Ministry of Education, with the co-operation of Tartu University, organized the first contest for senior high school students to solve problems of mathematics, physics, chemistry, astronomy and mechanics. Since 1954 annual national competitions, the so-called Estonian Olympiads on Exact Sciences (EOES), have been taking place. The main motivation for such competitions is to stimulate interest in exact sciences and to identify and support talented students in Estonia. The same time, the format of these contests has changed over the years.

At present in the framework of EOES three separate contests take place: olympiads in mathematics, in physics and in chemistry. Up to now, the olympiad languages have been Estonian and Russian. Every such olympiad consists of three rounds - at school, at county and at national level. The first round in mathematics for 5-12 form students usually is held in January. At school level the selection of problems for each form is made by teachers of mathematics.

The best students from 7-12 forms of each school are invited to the second round which usually takes place in February in the main center of this respective county. Problems both for this round as well as for the final round are proposed by the national jury. The administration and grading of the county round is done by teachers, and the best papers are sent to the national jury for evaluation. On the basis of county-round results, about fifty students from 9-12 forms in each subject will be invited in 1993 to the final round of the EOES. The final round is usually arranged in late March or in early April in Tartu University and it lasts for six days (two days for each subject). During 4-5 hours, students from each form solve 5 problems individually. The winners of EOES get popular mathematics books as prizes. In addition, ten best contestants from form 12 have the right to enter Estonian universities without taking any entrance examinations.

3. International contests

During 1960-1991, winners of the third round of EOES in Mathematics formed three-member teams of Estonia for the Mathematical Olympiad of USSR. During these years two Estonian students have shown top results and have been members of the IMO team of USSR.

In 1990 the team competition "Baltic Way" began, the format of which is the same as for the Lithuanian Team Mathematics Competition, i.e. each five member team from Estonia, Latvia and Lithuania solves twenty maths problems during five hours in separate rooms. The first such competition took place in Riga, the second in Tartu and this year's competition will take place in Vilnius.

In January this year two Estonian students from 9th form took part in the Mathematical Competition of Comprehensive Schools in Finland. Also, this year Estonia made its debute in IMO. It puts us under the obligation to improve the methods of preparing the Estonian team for IMO.

4. Special training

The expansion of mathematics competitions during the last decade is a stimulus in building up a new kind of mathematical education (see Laumen 1992). It is completely impossible to reach the international level for students in only one year of training if they come from a traditional educational system. We are sure that training in specialized groups encourages pupils to work harder in order to enrich their knowledge. It is especially important for students who are trained for participation in IMO that they would know everything, think quickly, operate and present their thoughts logically. However, the success in a competition depends on a specific set of skills which are consistent with but do not characterize the student's overall mathematical ability (see Barkeau 1989).

What would be the guarantee of success for students in olympiads? The answer seems to be very simple - every contestant must know

1) how to solve mathematical problems,

2) how all solutions must be presented, and

3) how to do it in the situation of competitions in a limited time?

Unfortunately it is difficult enough to put such requirements into practice. It is possible only through serious systematic training which should begin in form 7 at the least. Without doubt, teachers play an important role in this process. We must rely on teachers in the identification of their best pupils, in training their abilities in problem solving and in encouraging them to solve more complicated mathematical problems. The success in training of mathematically talented students is based on the teaching of mathematics in secondary schools and, in particular, in various forms of out-of-school activities.

5. Selection of students

The selection of students for the participation in training sessions to prepare them for IMO is now being started in Estonia. By EOES and some open contests every year 15-20 students will be selected as candidates for the IMO team. These students will be made familiar with mathematical and heuristic topics during three or four weekends, and one-week training session, which will be organized and administrated by the School of Exact Sciences for students in Tartu. The scientific supervision of all these activities for the young Estonian mathematicians is provided by the Faculty of Mathematics of Tartu University. Unfortunately, this kind of training work is limited both in the number of students and in time by scant financial resources. At present, some financial support has been obtained from our Ministry of Education.

It is sometimes observed that students (especially boys in forms up to 9) who do not excel in standard school mathematics courses, do well in mathematical problem solving competitions. On the other hand there are some able students in mathematics for whom olympiad-type problems are inappropriate. There are students who can solve very difficult problems but are not able to do it in a limited time. Therefore for the selection of contestants for IMO the experience of countries with a tradition of mathematical competitions should be taken into account.

6. Conclusion

Much knowledge and experience are necessary to achieve good results in olympiads. But knowledge and experience can be gained quickly by hard work, concentration and forethought, day and night (Bilchev & Rashkov 1991). Those of us who will be involved in contests have a special responsibility to ensure that the interests of the students remain central and that they do not become instruments of our own ambitions (see Barbeau 1989).

We do hope that the exchange of information and ideas on olympiads and training methods between students, teachers and mathematicians of Finland and Estonia which started this year will continue in future.

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On the Use of Graphs in Predicting the Level of Difficulty in Mathematical Items

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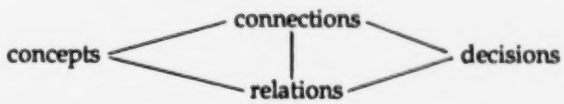
The following is a description of one approach to predicting the level of difficulty of mathematical items. The approach is based on the monograph "The logical structure of teaching material", written by the Ukrainian educationalist Dr A. Sohor.

1. The bases of A. Sohor's approach

In the above-mentioned monograph it is claimed that the level of difficulty of teaching material can be described extremely well by means of its so-called logical structure. In order to shape a logical structure it is necessary to solve the following problems:

1. The definition of invariants.
2. The choice of metalanguage.

A. Sohor defines the invariants of teaching material in the same way as the elements of *a logical structure*, as shown in the following diagram:



According to Sohor, graphs are best suited for converting invariants and the connections existing between them into metalanguage. These graphs he terms *structure patterns*.

An essential part of our research is also played by Sohor's assurance that '...the structure of an item is the structure of its solving procedure'. For this reason the starting point in designing the structure patterns of mathematical items is found within the optimal solving procedure. This significantly simplifies the complex process of designing structure patterns.

2. The graph and its characteristics as delineators of logical structure

According to Sohor the invariants in the item solving procedure are part of the structure pattern. Of these the following are suitable as the *nodes* of the graph :

- initial data
- initial conditions
- results
- definitions
- theorems
- rules
- instructions, etc.

The *sides of the graph* describe the connections, relations and application order existing between these.

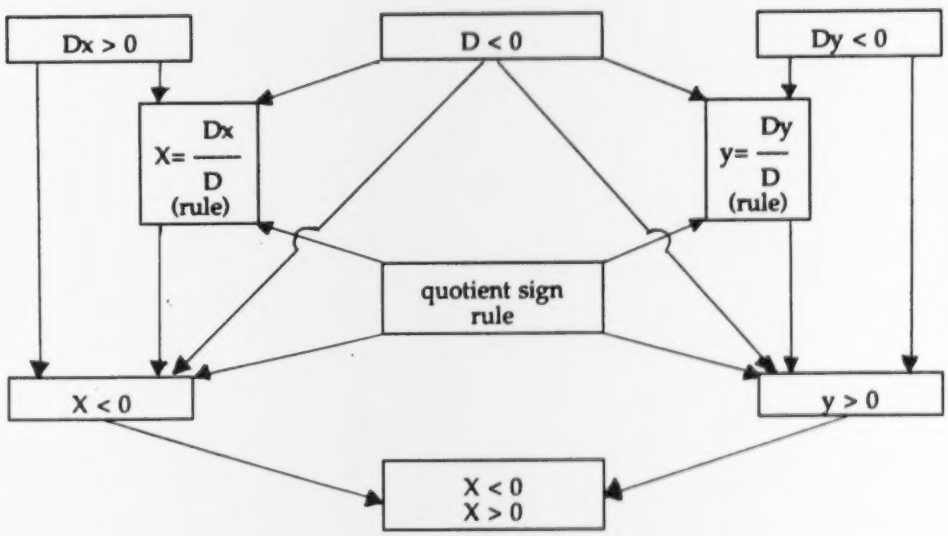
The most contentious question in designing a graph is the elementariness of the invariants to be used. It primarily depends on the developmental level of the solver, the pupil. Our experience suggests that the definition of elements requires careful consideration, but that it nevertheless remains at the level of estimation.

As an example, we present the structure patterns for a 7th-grade algebra item.

Item. In the linear equation pair $D < O$, $Dx > O$ and $Dy < O$, which of the following is valid?

A) $\begin{cases} x < 0 \\ y < 0 \end{cases}$ B) $\begin{cases} x < 0 \\ y > 0 \end{cases}$ C) $\begin{cases} x > 0 \\ y > 0 \end{cases}$ D) $\begin{cases} x > 0 \\ y < 0 \end{cases}$

Structure pattern



A. Sohor claims that of all the possible characteristics of the graph the following six best describe the complexity of the structure:

1. The number of nodes (in our example, n=9)
2. The number of sides (m=16)
3. Periodicity value (m-n+1=8)
4. The number of nodes belonging to enclosed frames (47)
5. The number of nodes on the longest route on the graph (9)
6. The ratio $p = \frac{2m}{n} = 3.6$

3. Description of experiment and main results

The purpose was to evaluate how well the six characteristics of the structure patterns (SP) in school maths items designed according to Sohor's theory correlated with the actual level of difficulty (LD) of the items. The level of difficulty was evaluated by means of multiple-choice tests which revealed the LD of the items as a percentage of correct answers. A strong negative correlation between SP and LD would indicate that the SP is a good predictor of the level of difficulty of the item. This prediction could be used in designing learning and testing material.

140 mathematical items of the 7th grade were examined. They were divided

into seven tests with 10 items in each test and two parallel variants for each item. The subjects were some 300 pupils from 13 Estonian schools. The SPs for the entire 140 items were written by three members of a workgroup under the direction of the writer.

We also examined the extent to which the SP of the items depended on the item designer. For this reason a second parallel two-person workgroup was formed which also wrote SPs for the items using the same approaches and conditions. The correlations between the SP series of the two groups are shown in Table 1.

TABLE 1. *Correlations between the SP values of the evaluation groups*

SP characteristic	1	2	3	4	5	6
Correlation	.88	.84	.72	.63	.71	.89

The reliability levels of the SP characteristics drawn up on the basis of the table are at the very least satisfactory.

The correlations between the SP characteristics (1. - 6.) of all 140 items and their levels of difficulty (LD) are shown in Table 2.

TABLE 2. *Correlations between characteristics and levels of difficulty*

1	2	3	4	5	6	n = 140
-0,23	-0,22	-0,26	-0,33	-0,30	-0,21	LD
	0,99	0,94	0,91	0,79	0,82	1
		0,97	0,94	0,76	0,84	2
			0,94	0,66	0,84	3
				0,67	0,90	4
					0,68	5

The characteristics of the structure formula do not generally correlate highly with the actual level of difficulty of the items. The correlation of the level of difficulty was somewhat higher with the figures indicating nodes forming enclosed frames (r=-0.33) and nodes of the longest route (r=-0.30). The internal SP correlations were very high.

We also examined whether the correlations depended on the item domain. To this end the items were divided into algebra, arithmetic, geometry and spatial geometry items. The highest correlations were found for algebra items. This is also

natural since the solving procedure of algebra items is primarily an algorithmic process.

4. Summary

Our examination of A. Sohor's theory for defining the level of difficulty of mathematical items does not yield sufficiently good results at the present level of development of the theory The foundations and technology for designing a structure model require considerable additions and improvements.

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ISBN 951-34-0180-4
ISSN 0782-9817